Math 75B Practice Midterm I – Solutions
Ch. 9-11, 14-C.2 (Ebersole), §§2.6, 2.7, 3.3, 3.5, 3.7 (Stewart)

DISCLAIMER. This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

True or False. Circle **T** if the statement is *always* true; otherwise circle **F**.

1. If $g(x) = x^x$, then $g'(x) = x \cdot x^{x-1}$. **T** $\boxed{\textbf{F}}$

   The power rule does not apply here. If you use logarithmic differentiation, you will get $g'(x) = (1 + \ln x)x^x$.

2. $\dfrac{d}{dt}\cos^{-1}(3t^2+1) = -\dfrac{6t}{\sqrt{1-(3t^2+1)^2}}$. $\boxed{\textbf{T}}$ **F**

   Using the chain rule, we have $\dfrac{d}{dt}\cos^{-1}(3t^2+1) = -\dfrac{1}{\sqrt{1-(3t^2+1)^2}} \cdot 6t = -\dfrac{6t}{\sqrt{1-(3t^2+1)^2}}$.

3. $\dfrac{d}{dt}\tan^{-1}(3t^2+1) = \dfrac{6t}{\sqrt{1-(3t^2+1)^2}}$. **T** $\boxed{\textbf{F}}$

   The derivative of $\tan^{-1} t$ is $\dfrac{1}{1+t^2}$, so using the chain rule the derivative of $\tan^{-1}(3t^2+1)$ is $\dfrac{1}{1+(3t^2+1)^2} \cdot (6t) = \dfrac{6t}{1+(3t^2+1)^2}$.

4. The range of the function $f(x) = \sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. $\boxed{\textbf{T}}$ **F**

   This is the proper restricted domain for the sine function $\sin(x)$, which is also the range of the inverse sine function $\sin^{-1}(x)$.

5. The range of the function $f(x) = \cos^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. **T** $\boxed{\textbf{F}}$

   The range of $\cos^{-1}(x)$ is $[0, \pi]$.

6. $\dfrac{d}{dt}(\log(3t^2+1)) = \dfrac{(6\ln 10)t}{3t^2+1}$. **T** $\boxed{\textbf{F}}$

   The derivative of $\log t$ (when the base is not shown it is the logarithm with base 10) is $\dfrac{1}{\ln 10 \cdot t}$, so using the chain rule the derivative of $\log(3t^2+1)$ is $\dfrac{1}{\ln 10 \cdot (3t^2+1)} \cdot (6t) = \dfrac{6t}{\ln 10(3t^2+1)}$.

7. If $3x^2y = \tan(y^2)$, then $\dfrac{dy}{dx} = \dfrac{-6xy}{3x^2 - 2y\sec^2(y^2)}$. **[T]** **F**

Taking the derivative of both sides implicitly with respect to x (don't forget the product rule on the left side!), we have

$$\begin{aligned} 3x^2\, y' + y \cdot 6x &= \sec^2(y^2) \cdot 2y\, y' \\ 3x^2\, y' - \sec^2(y^2) \cdot 2y\, y' &= -6xy \\ (3x^2 - 2y\sec^2(y^2))y' &= -6xy \\ y' &= \frac{-6xy}{3x^2 - 2y\sec^2(y^2)}. \end{aligned}$$

8. The limit $\lim_{x\to 0} \left(\dfrac{1}{x}\right)^{\cos x}$ is an indeterminate form. **T** **[F]**

When we "plug in infinity," we get "∞^1", which is always ∞.

Multiple Choice. *Circle the letter of the best answer.*

1. If $x^2 - y^2 = 4$, then $\dfrac{dy}{dx} =$

(a) $\dfrac{y}{x}$

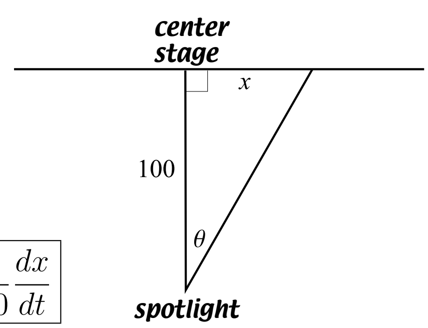
(b) $-\dfrac{y}{x}$

(c) $\boxed{\dfrac{x}{y}}$

(d) $-\dfrac{x}{y}$

We have $2x - 2y\, y' = 0$, so solving for y' we get $2y\, y' = 2x$, or $y' = \dfrac{dy}{dx} = \dfrac{2x}{2y} = \dfrac{x}{y}$.

2. A ballet dancer tiptoes across a stage while a spotlight 100 ft. away from center stage follows her, as shown. An equation to describe the relationship between the rate of change of the angle θ and the dancer's speed is



(a) $\sec\theta \cdot \dfrac{d\theta}{dt} = \dfrac{100}{x}\dfrac{dx}{dt}$

(b) $\tan\theta = \dfrac{x}{100}$

(c) $\cos\theta \cdot \dfrac{d\theta}{dt} = \dfrac{1}{100}$

(d) $\boxed{\sec^2\theta \cdot \dfrac{d\theta}{dt} = \dfrac{1}{100}\dfrac{dx}{dt}}$

From the picture, an equation to describe the relationship between x and θ is $\tan\theta = \dfrac{x}{100}$. Taking the derivative of both sides with respect to t, we get $\sec^2\theta \cdot \dfrac{d\theta}{dt} = \dfrac{1}{100}\dfrac{dx}{dt}$.

3. $\ln\left(\dfrac{e^x\sqrt{x}}{(5x-1)^{\cos x}}\right) =$

(a) $\boxed{x - \frac{1}{2}\ln x - \cos x \ln(5x-1)}$

(b) $\dfrac{(5x-1)^{\cos x}}{e^x\sqrt{x}}$

(c) $\dfrac{\ln\left(x - \frac{1}{2}\ln x\right)}{\ln(\cos x \ln(5x-1))}$

(d) $\ln\left(\dfrac{\frac{1}{2}e^x x^{-1/2}}{-5\sin x(5x-1)^{\cos x}}\right)$

Using logarithm laws, we know that

$$\begin{aligned}\ln\left(\frac{e^x\sqrt{x}}{(5x-1)^{\cos x}}\right) &= \ln\left(e^x\sqrt{x}\right) - \ln\left((5x-1)^{\cos x}\right)\\ &= \ln(e^x) + \ln(\sqrt{x}) - \ln\left((5x-1)^{\cos x}\right)\\ &= x + \frac{1}{2}\ln x - (\cos x)\ln(5x-1)\end{aligned}$$

If you are careful you can try to use all three logarithm laws together to do this problem in a single step. Just be careful exactly what goes where!

4. If $f(x) = \sqrt[5]{\sin^{-1}(e^x)}$, then $f'(x) =$

(a) $\dfrac{1}{5}(\sin^{-1}(e^x))^{-4/5}$

(b) $\boxed{\dfrac{e^x(\sin^{-1}(e^x))^{-4/5}}{5\sqrt{1-e^{2x}}}}$

(c) $\dfrac{e^x}{5\sqrt{1-e^{2x}}}$

(d) $\dfrac{1}{5}\left(\dfrac{e^x}{\sqrt{1-e^{2x}}}\right)^{-4/5}$

Using the chain rule a couple times, we have

$$\begin{aligned}f'(x) &= \frac{1}{5}(\sin^{-1}(e^x))^{-4/5}\cdot\frac{1}{\sqrt{1-(e^x)^2}}\cdot e^x\\ &= \frac{e^x(\sin^{-1}(e^x))^{-4/5}}{5\sqrt{1-e^{2x}}}.\end{aligned}$$

5. The limit $\lim\limits_{x\to 3}(\ln(x-3))^{x^2-9}$ is an indeterminate form of type

(a) $\boxed{\infty^0}$

(b) 0^0

(c) 1^∞

(d) none; not an indeterminate form

This is what you get when you try to "plug in 3".

6. $\lim_{x\to 1^+} x^{\left(\frac{1}{\ln x}\right)} =$

(a) 1
(b) 0
(c) $\boxed{e}$
(d) does not exist

This is an indeterminate form of type 1^∞, so we use a logarithm to evaluate the limit. We have

$$\begin{aligned} L &= \lim_{x\to 1^+} x^{\left(\frac{1}{\ln x}\right)} \\ \ln(L) &= \ln\left(\lim_{x\to 1^+} x^{\left(\frac{1}{\ln x}\right)}\right) \\ &= \lim_{x\to 1^+} \ln\left(x^{\left(\frac{1}{\ln x}\right)}\right) \\ &= \lim_{x\to 1^+} \left(\frac{1}{\ln x}\right)\ln x \\ &= \lim_{x\to 1^+} 1 = 1. \end{aligned}$$

Therefore the original limit is $L = e^1 = e$.

Fill-In.

1. $\sin^{-1}(1) =$ $\underline{\frac{\pi}{2}}$

$\sin\left(\frac{\pi}{2}\right) = 1$. Since $\frac{\pi}{2}$ is in the correct range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, this is the correct answer.

2. $\cos^{-1}(1) =$ $\underline{0}$

$\cos(0) = 1$. Since 0 is in the correct range $[0, \pi]$, this is the correct answer.

3. $\sin^{-1}(\cos 3\pi) =$ $\underline{-\frac{\pi}{2}}$

$\sin^{-1}(\cos 3\pi) = \sin^{-1}(-1) = -\frac{\pi}{2}$.

4. $\cos^{-1}\left(\tan\frac{\pi}{4}\right) =$ $\underline{0}$

$\cos^{-1}\left(\tan\frac{\pi}{4}\right) = \cos^{-1}(1) = 0$.

5. $\tan\left(\cos^{-1}\frac{2}{7}\right) =$ $\underline{\frac{\sqrt{45}}{2}}$

Draw a right triangle showing an angle θ whose cosine is $\frac{2}{7}$ (adjacent leg is 2, hypotenuse is 7). Use the Pythagorean Theorem to figure out what the third side of the triangle must be (you should get $\sqrt{45}$). Then note that the tangent of θ is $\frac{\sqrt{45}}{2}$.

6. $\sin\left(\sin^{-1}\frac{5}{2}\right) =$ $\underline{\text{undefined}}$

There is no angle whose sine is $\frac{5}{2}$.

7. $\sin^{-1}\left(\sin\frac{3\pi}{2}\right) =$ $\underline{-\frac{\pi}{2}}$

$\sin^{-1}\left(\sin\frac{3\pi}{2}\right) = \sin^{-1}(-1) = -\frac{\pi}{2}$.

Work and Answer. *You must show all relevant work to receive full credit.*

1. If $y \tan y = 3t - \dfrac{y}{t}$, find $\dfrac{dy}{dt}$.

   Taking the derivative of both sides implicitly with respect to t, we have

$$y \sec^2 y \cdot y' + \tan y \cdot y' = 3 - \frac{t\,y' - y}{t^2};$$

   solving for $y' = \dfrac{dy}{dt}$, we get

$$\begin{aligned} y \sec^2 y \cdot y' + \tan y \cdot y' &= 3 - \frac{t\,y'}{t^2} + \frac{y}{t^2} \\ y \sec^2 y \cdot y' + \tan y \cdot y' + \frac{1}{t}\,y' &= 3 + \frac{y}{t^2} \\ \left(y \sec^2 y + \tan y + \frac{1}{t} \right) y' &= 3 + \frac{y}{t^2} \\ y' &= \boxed{\frac{3 + \frac{y}{t^2}}{y \sec^2 y + \tan y + \frac{1}{t}}} \end{aligned}$$

2. Find the slope of the tangent line to the graph of $\dfrac{(x+2)^2}{9} + \dfrac{y^2}{4} = 1$ at the point $(-2, 2)$.

   Taking the derivative of both sides implicitly with respect to x, we have

$$\frac{2}{9}(x+2) + \frac{1}{2}y \cdot \frac{dy}{dx} = 0.$$

   Plugging in $x = -2$, $y = 2$, we have

$$0 + \frac{1}{2} \cdot 2 \cdot \frac{dy}{dx} = 0,$$

   so solving for $\dfrac{dy}{dx}$ we get $\dfrac{dy}{dx} = 0$.
   As a reality check, the equation represents an ellipse centered at $(-2, 0)$ whose highest point is $(-2, 2)$, so the tangent line is horizontal there.

3. Find the derivative of the function $g(x) = (\sin x)^{3x+1}$.

   Using logarithmic differentiation, we have

$$\begin{aligned} \ln g(x) &= \ln\left((\sin x)^{3x+1}\right) \\ \ln g(x) &= (3x+1)\ln(\sin x) \\ \frac{g'(x)}{g(x)} &= (3x+1) \cdot \frac{\cos x}{\sin x} + \ln(\sin x) \cdot 3 \\ \frac{g'(x)}{g(x)} &= (3x+1)\cot x + 3\ln(\sin x) \\ g'(x) &= \boxed{\left((3x+1)\cot x + 3\ln(\sin x)\right)(\sin x)^{3x+1}} \end{aligned}$$

4. Find the derivative of the function $h(x) = \dfrac{(3x^2-4)^{10}\cos(4x)}{e^x(59x^3+8x)^{25}}$.

The easiest way to do this problem is by logarithmic differentiation. We have

$$\ln(h(x)) = 10\ln(3x^2-4) + \ln(\cos(4x)) - x - 25\ln(59x^3+8x)$$

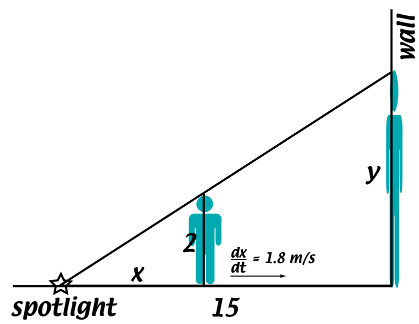
(I'm skipping several steps here; you should convince yourself, one step at a time if necessary, that you agree with the above — or you should let me know if I have made a mistake!). Now we are ready to take the derivative of both sides. We have

$$\frac{h'(x)}{h(x)} = \frac{10\cdot 6x}{3x^2-4} + \frac{-\sin(4x)\cdot 4}{\cos(4x)} - 1 - \frac{25(177x^2+8)}{59x^3+8x}$$

$$\frac{h'(x)}{h(x)} = \frac{60x}{3x^2-4} - 4\tan x - 1 - \frac{25(177x^2+8)}{59x^3+8x}$$

$$h'(x) = \boxed{\left(\frac{60x}{3x^2-4} - 4\tan x - 1 - \frac{25(177x^2+8)}{59x^3+8x}\right)\frac{(3x^2-4)^{10}\cos(4x)}{e^x(59x^3+8x)^{25}}}$$

5. A spotlight on the ground shines on a wall 15 m away. If a man 2 m tall walks from the spotlight toward the wall at a speed of 1.8 m/s, how fast is the length of his shadow on the wall decreasing when he is 3 m from the building?



From the picture, we set up similar triangles to get the equation $\frac{2}{x} = \frac{y}{15}$, which can be rewritten as

$$2x^{-1} = \frac{1}{15}y$$

to make it more "derivative-friendly."

Now take the derivative of both sides. We get

$$-\frac{2}{x^2}\cdot\frac{dx}{dt} = \frac{1}{15}\frac{dy}{dt}.$$

Note that when the man is 3 m from the building, $x = 12$. We can also plug in $\frac{dx}{dt} = 1.8$. Solving for $\frac{dy}{dt}$ we have

$$-\frac{2}{12^2}\cdot(1.8) = \frac{1}{15}\frac{dy}{dt}$$

$$\frac{dy}{dt} = -\frac{15\cdot 2\cdot 1.8}{144} = -\frac{3}{8}.$$

Therefore the length of his shadow is $\boxed{\text{decreasing at a rate of } \frac{3}{8} \text{ m/s}}$

6. Find the limit $\lim_{t\to 0^+} \ln t \sin t$.

First recall: we proved in class that

$$\lim_{t\to 0} \frac{\sin t}{t} = 1 \tag{1}$$

(you can also verify this using l'Hôpital's Rule). This fact will be useful in this problem. The limit $\lim_{t\to 0^+} \ln t \sin t$ is an indeterminate form of type $0 \cdot (-\infty)$. We have

$$\begin{aligned}
\lim_{t\to 0^+} \ln t \sin t &= \lim_{t\to 0^+} \frac{\ln t}{\csc t} \\
&\overset{\mathrm{H}}{=} \lim_{t\to 0^+} \frac{\frac{1}{t}}{-\csc t \cot t} \\
&= \lim_{t\to 0^+} -\frac{\sin t \tan t}{t} \\
&= \lim_{t\to 0^+} \frac{\sin t}{t} \cdot \lim_{t\to 0^+} (-\tan t) \qquad \text{(using properties of limits)} \\
&= 1 \cdot 0 \qquad \text{(by (1) above)} \\
&= \boxed{0}
\end{aligned}$$

7. Find the limit $\lim_{x\to 0^+} (\sin^{-1} x)^x$.

This is an indeterminate form of type 0^0, so we use a logarithm to evaluate the limit. We have

$$\begin{aligned}
L &= \lim_{x\to 0^+} (\sin^{-1} x)^x \\
\ln(L) &= \ln\left(\lim_{x\to 0^+} (\sin^{-1} x)^x\right) \\
&= \lim_{x\to 0^+} \ln\left((\sin^{-1} x)^x\right) = \lim_{x\to 0^+} x \ln(\sin^{-1} x)
\end{aligned}$$

This is now an indeterminate product of type $0 \cdot (-\infty)$. So we rewrite it as a fraction in order to be able to use l'Hoṗital's Rule (we'll have to use it twice):

$$\begin{aligned}
&= \lim_{x\to 0^+} \frac{\ln(\sin^{-1} x)}{\frac{1}{x}} \qquad \left(\text{“}-\tfrac{\infty}{\infty}\text{”}\right) \\
&\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{(\sin^{-1} x)\sqrt{1-x^2}}}{-\frac{1}{x^2}} \\
&= \lim_{x\to 0^+} \frac{-x^2}{(\sin^{-1} x)\sqrt{1-x^2}} \qquad \left(\text{“}\tfrac{0}{0}\text{”}\right) \\
&\overset{H}{=} \lim_{x\to 0^+} \frac{-2x}{\frac{-2x\sin^{-1} x}{2\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}} \\
&= \lim_{x\to 0^+} \frac{-2x}{\frac{-x\sin^{-1} x}{\sqrt{1-x^2}} + 1} = \frac{0}{\frac{0}{1} + 1} = 0.
\end{aligned}$$

Therefore the original limit is $L = e^0 = \boxed{1}$