Math 75B Practice Problems for Midterm III – Solutions

True or False. Circle **T** if the statement is *always* true; otherwise circle **F**.

1. If the velocity of an object at time t is $v(t) = 4t^2 + 1$ ft./s, then its distance in feet at time t is $s(t) = \frac{4}{3}t^3 + t$. **T** $\boxed{\textbf{F}}$

   The distance might be $s(t) = \frac{4}{3}t^3 + t + 450$. There is no initial condition in the problem. Another way to put this is, the problem does not say "distance *from* somewhere." Maybe $s(t)$ represents the distance from Egypt! Since we don't know, we can't say for sure which antiderivative represents the distance.

2. The function $F(x) = \sin 2x + 52$ is an antiderivative of the function $f(x) = 2\cos 2x$. $\boxed{\textbf{T}}$ **F**

   We can check this by taking the derivative of $F(x)$. We get $F'(x) = \cos 2x \cdot 2 = 2\cos 2x = f(x)$.

3. The function $G(x) = 4x^3$ is an antiderivative of the function $g(x) = x^4 - 2$. **T** $\boxed{\textbf{F}}$

   $G'(x) = 12x^2 \neq g(x)$.

4. $-1 + 0 + \frac{1}{3} + \frac{1}{2} + \frac{3}{5} + \frac{2}{3} = \sum_{i=-1}^{4} \frac{i}{i+2}$. $\boxed{\textbf{T}}$ **F**

   This is a tricky problem, because the pattern on the left-hand side of the equation is disguised. However, all we have to do is find the sum on the right and see if it simplifies to the sum on the left:

$$\begin{aligned}\sum_{i=-1}^{4} \frac{i}{i+2} &= \frac{-1}{1} + \frac{0}{2} + \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} \\ &= -1 + 0 + \frac{1}{3} + \frac{1}{2} + \frac{3}{5} + \frac{2}{3}.\end{aligned}$$

   Sure enough, the equation is valid.

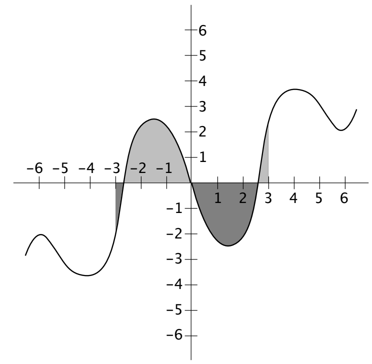
5. $\sum_{i=2}^{4} \frac{i^2}{2} = \frac{29}{2}$. $\boxed{\textbf{T}}$ **F**

   We have

$$\sum_{i=2}^{4} \frac{i^2}{2} = \frac{2^2}{2} + \frac{3^2}{2} + \frac{4^2}{2} = \frac{4+9+16}{2} = \frac{29}{2}.$$

6. If $g(x)$ is an odd function which is continuous on the interval $[-3,3]$, then $\int_{-3}^{3} g(x)\,dx = 0$. $\boxed{\textbf{T}}$ **F**

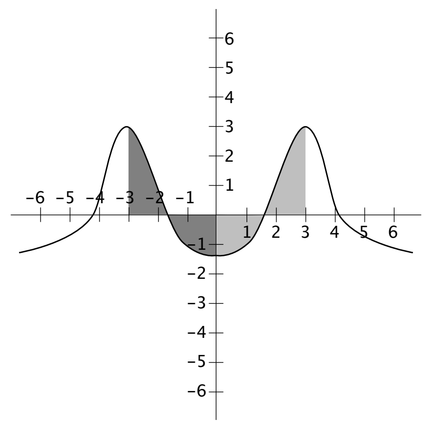
If $g(x)$ is an odd function, that means the graph of $g(x)$ is symmetric about the origin. Therefore between -3 and 3 there is the same amount of area above the x-axis as below. (An example is shown at right.)



7. If $h(x)$ is an even function which is continuous on the interval $[-3,3]$, then $\int_{-3}^{3} h(x)\,dx = 2\int_{0}^{3} h(x)\,dx$. $\boxed{\textbf{T}}$ **F**

This is also true! If $h(x)$ is an even function, that means the graph of $h(x)$ is symmetric about the y-axis. Therefore whatever area there is between 0 and 3 there is the same amount of area between -3 and 0. (An example is shown at right.) So we can just double the amount we get from 0 to 3.

This can be a very useful fact when doing definite integrals, since 0 is a lot easier to plug in than -3.



Multiple Choice. *Circle the letter of the best answer.*

1. If $x_1 = 1$ is a first approximation of a solution to the equation $x^4 = 6 - 3x$, then using Newton's Method the second approximation is $x_2 =$

$\boxed{\text{(a)}}$ $\frac{9}{7}$

(b) $\frac{5}{7}$

(c) $\frac{9}{2}$

(d) $-\frac{5}{2}$

First we let $f(x) = x^4 + 3x - 6$. Then finding a solution to the equation $x^4 = 6 - 3x$ is the same as finding a root of $f(x)$. The formula for x_2 is $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$. Now $f(1) = 1 + 3 - 6 = -2$ and $f'(x) = 4x^3 + 3$, so $f'(1) = 7$, and we get

$$\begin{aligned} x_2 &= 1 - \frac{f(1)}{f'(1)} \\ &= 1 - \frac{-2}{7} \\ &= 1 + \frac{2}{7} = \boxed{\frac{9}{7}}. \end{aligned}$$

2. $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\, dx =$

(a) $-\dfrac{1}{6}$

(b) 0

$\boxed{\text{(c)}}$ 2π

(d) does not exist.

The curve $y = \sqrt{4-x^2}$ is the upper half of a circle with radius 2 centered at $(0,0)$. Therefore the integral $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\, dx$ represents the area of a semicircle of radius 2, which is $\dfrac{1}{2} \cdot \pi \cdot 2^2 = \boxed{2\pi}$.

3. $\displaystyle\int_{0}^{\pi/4} \sec x \tan x \, dx =$

$\boxed{\text{(a)}}$ $\sqrt{2}-1$

(b) $\sqrt{2}$

(c) $1 - \dfrac{\sqrt{2}}{2}$

(d) does not exist.

The function $f(x) = \sec x \tan x$ is defined on the interval $\left[0, \dfrac{\pi}{4}\right]$, so the integral exists. Using the Fundamental Theorem of Calculus we have

$$\begin{aligned} \int_{0}^{\pi/4} \sec x \tan x \, dx &= \sec x \Big|_{0}^{\pi/4} \\ &= \sec\left(\frac{\pi}{4}\right) - \sec(0) \\ &= \boxed{\sqrt{2}-1} \end{aligned}$$

4. If $f(x)$ is continuous, then $\displaystyle\int_{0}^{\pi/4} f(x)\, dx - \int_{0}^{\pi} f(x)\, dx =$

(a) $\displaystyle\int_{\pi/4}^{\pi} f(x)\, dx$

$\boxed{\text{(b)}}$ $\displaystyle\int_{\pi}^{\pi/4} f(x)\, dx$

(c) $\displaystyle\int_{0}^{3\pi/4} f(x)\, dx$

(d) $\displaystyle\int_{\pi}^{0} f(x)\, dx$

This is just the property $\displaystyle\int_{a}^{c} f(x)\, dx - \int_{a}^{b} f(x)\, dx = \int_{b}^{c} f(x)\, dx.$

5. If $u = 3x^2 - 5$, then the integral $\displaystyle\int x\cos^2(3x^2-5)\, dx$ is equivalent to

(a) $\displaystyle\int u\cos^2(u)\, du$

(b) $\displaystyle\frac{1}{2}\int \cos^2(u)\, du$

(c) $\displaystyle\frac{1}{3}\int \cos^2(u)\, du$

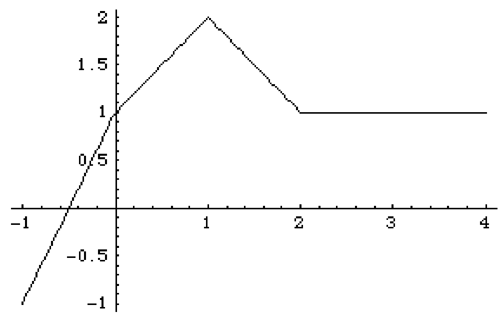
$\boxed{\text{(d)}}$ $\displaystyle\frac{1}{6}\int \cos^2(u)\, du$

We have $du = 6x\, dx$, therefore by "futzing" the constant multiple 6 we have

$$\int x\cos^2(3x^2-5)\, dx = \frac{1}{6}\int 6x\cos^2(3x^2-5)\, dx = \frac{1}{6}\int \cos^2(u)\, du.$$

For #6-7, use the graph of $f(x)$ shown below to answer the questions.

6. $\displaystyle\int_{-1}^{0} f(x)\,dx =$



(a) -1

(c) 1

$\boxed{\text{(b)}}$ 0

(d) 2

Between $x = -1$ and $x = 0$ there is just as much area (between $f(x)$ and the x-axis) above the x-axis as below. Therefore the net area is $\boxed{0}$.

7. $\displaystyle\int_{2}^{1} f(x)\,dx =$

$\boxed{\text{(a)}}$ $-\dfrac{3}{2}$

(c) -1

(b) $\dfrac{3}{2}$

(d) 1

The area under $f(x)$ from $x = 1$ to $x = 2$ is $\dfrac{3}{2}$. But the limits of integration have been reversed. So

$$\int_{2}^{1} f(x)\,dx = -\int_{1}^{2} f(x)\,dx = \boxed{-\frac{3}{2}}$$

Fill-In. *If an answer is undefined, write "D.N.E."*

1. $\displaystyle\int (\sqrt[3]{x} - \sec^2 x)\,dx = \underline{\ \frac{3}{4}x^{4/3} - \tan x + C\ }$.

$\sqrt[3]{x} = x^{1/3}$, so $\displaystyle\int (\sqrt[3]{x} - \sec^2 x)\,dx = \tfrac{3}{4}x^{4/3} - \tan x + C$ (this is straight out of the formulas in Stewart).

2. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \underline{\ \sin^{-1}(x) + C\ }$.

This is also straight out of the formulas in Stewart, since we learned at the beginning of the term that $\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$.

3. $\displaystyle\int_{-1}^{2} \sqrt[4]{x}\,dx = \underline{\text{ D.N.E. }}$

$\sqrt[4]{x}$ is undefined for $x < 0$. Therefore the integral does not make sense.

4. $\displaystyle\int_{-1}^{2} \sqrt[3]{x}\, dx = \underline{\frac{3}{4}\left(2\sqrt[3]{2} - 1\right)}$

Here we have no domain problems — $\sqrt[3]{x}$ is defined for all x. So we use F.T.C. (Fundamental Theorem of Calculus):

$$\begin{aligned}
\int_{-1}^{2} \sqrt[3]{x}\, dx &= \int_{-1}^{2} x^{1/3}\, dx \\
&= \frac{3}{4} x^{4/3} \Big|_{-1}^{2} \\
&= \frac{3}{4}\left(2^{4/3} - (-1)^{4/3}\right) \\
&= \frac{3}{4}\left(2\sqrt[3]{2} - 1\right).
\end{aligned}$$

You can also leave your answer as $\frac{3}{4}\left(2^{4/3} - 1\right)$ or anything equal to it. Simplify whenever you can, but *at your own risk!*

5. $\displaystyle\int 3e^{5x}\, dx = \underline{\frac{3}{5}e^{5x} + C}$.

You can do this integral by the "guess and check" method or by a formal u-substitution (use $u = 5x$).

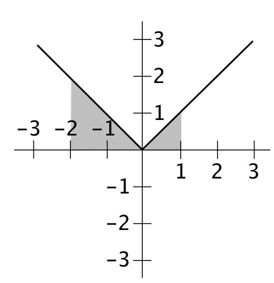
6. $\displaystyle\int \cos(3x - 5)\, dx = \underline{\frac{1}{3}\sin(3x - 5) + C}$.

You can do this integral by the "guess and check" method or by a formal u-substitution (use $u = 3x - 5$).

7. If the interval $[-4, 7]$ is divided into 6 equal subintervals, then the width of each subinterval is $\underline{\frac{11}{6}}$.

The interval $[-4, 7]$ is 11 ($= 7 - (-4)$) units wide. If we divide 11 units into 6 equal parts, each part will be $\frac{11}{6}$ units wide.

8. $\displaystyle\int_{-2}^{1} |x|\, dx = \underline{\frac{5}{2}}$.



The easiest way to do this problem is to think about **areas**. The graph of $|x|$, with the area from -2 to 1 shaded, is shown at right.

Notice that we have two triangles. The areas of the triangles are $\frac{2 \cdot 2}{2} = 2$ and $\frac{1 \cdot 1}{2} = \frac{1}{2}$. Since they are both above the x-axis, we add:

$$\int_{-2}^{1} |x|\, dx = 2 + \frac{1}{2} = \boxed{\frac{5}{2}}$$

9. If $F(x) = \int_5^x \sqrt{5t - t^4}\, dt$, then $F'(x) = \underline{\ \sqrt{5x - x^4}\ }$.

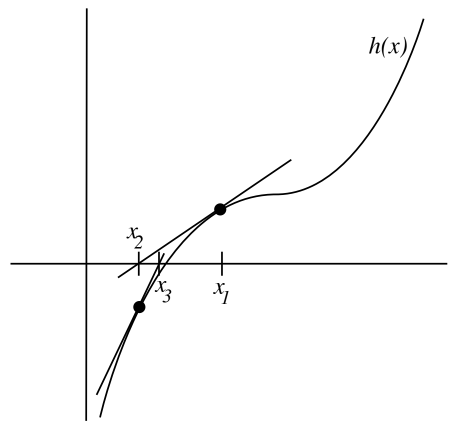
   This is exactly what F.T.C. Part 1 says.

10. If $G(x) = \int_x^{\sqrt{\pi}} \cot(6t^2)\, dt$, then $G'(x) = \underline{\ -\cot(6x^2)\ }$.

    We have $G(x) = \int_x^{\sqrt{\pi}} \cot(6t^2)\, dt = -\int_{\sqrt{\pi}}^x \cot(6t^2)\, dt$, so using F.T.C. Part 1 we get the answer shown (the opposite to what we would have gotten if the x had been in the upper limit in the original problem).

Graph.
For the function $h(x)$ graphed at right and the initial guess x_1 shown, draw tangent lines to determine x_2 and x_3 according to Newton's Method, as in the sample (the sample is only completed through x_2). Label x_2 and x_3 on the x-axis.

Newton's Method takes each guess, finds the equation of the tangent line to $f(x)$ at that x-value, and gets the x-intercept of that line, which becomes the next guess. You can see how this is graphed at right.



Work and Answer. *You must show all relevant work to receive full credit.*

1. Estimate the root of $f(x) = x^3 + 2x - 1$ using two iterations of Newton's Method (*i.e.* compute x_3) with the initial guess $x_1 = 0$. *Express your answer as an exact fraction.*

   We use the formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ twice. $f'(x) = 3x^2 + 2$, so we get

$$x_2 = 0 - \frac{f(0)}{f'(0)} \qquad f(0) = -1 \qquad f'(0) = 2$$

$$= 0 - \frac{-1}{2} = \frac{1}{2}\,.$$

$$x_3 = \frac{1}{2} - \frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)} \qquad f\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^3 + 2\left(\tfrac{1}{2}\right) - 1 = \tfrac{1}{8}$$

$$= \frac{1}{2} - \frac{\frac{1}{8}}{\frac{11}{4}} \qquad f'\left(\tfrac{1}{2}\right) = 3\left(\tfrac{1}{2}\right)^2 + 2 = \tfrac{3}{4} + 2 = \tfrac{11}{4}$$

$$= \frac{1}{2} - \frac{4}{8\cdot 11}$$

$$= \frac{1}{2} - \frac{1}{22} = \frac{11-1}{22} = \frac{10}{22} = \boxed{\frac{5}{11}}\,.$$

2. Evaluate $\int \frac{2}{t^3}\,dt$.

We have

$$\begin{aligned}\int \frac{2}{t^3}\,dt &= \int 2t^{-3}\,dt \\ &= 2\cdot\frac{1}{-2}t^{-2}+C = -t^{-2}+C = \boxed{-\frac{1}{t^2}+C}\end{aligned}$$

3. Evaluate $\int \frac{2}{1+x^2}\,dx$.

Since $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$, we have $\int \frac{2}{1+x^2}\,dx = \boxed{2\tan^{-1}(x)+C}$

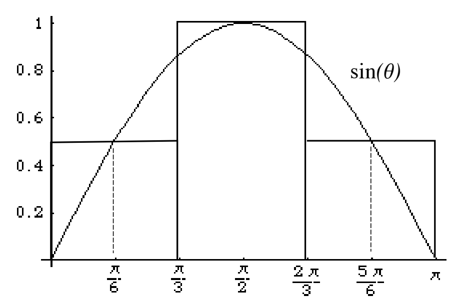
4. Evaluate $\int_0^{\pi/3} x^2 - \sin x\,dx$.

The function $f(x) = x^2 - \sin x$ is continuous on the interval $\left[0, \frac{\pi}{3}\right]$, so the integral makes sense. We have

$$\begin{aligned}\int_0^{\pi/3} x^2 - \sin x\,dx &= \frac{1}{3}x^3 + \cos x\,\Big|_0^{\pi/3} \\ &= \left(\frac{1}{3}\left(\frac{\pi}{3}\right)^3 + \cos\left(\frac{\pi}{3}\right)\right) - (0+1) \\ &= \boxed{\frac{\pi^3}{81}+\frac{1}{2}-1}\end{aligned}$$

5. (a) <u>Estimate</u> $\int_0^{\pi} \sin\theta\,d\theta$ using 3 rectangles and midpoints.

We are dividing the interval $[0,\pi]$ into 3 equal parts, so each part is of width $\Delta x = \frac{b-a}{n} = \frac{\pi-0}{3} = \frac{\pi}{3}$. So the three subintervals are $\left[0,\frac{\pi}{3}\right]$, $\left[\frac{\pi}{3},\frac{2\pi}{3}\right]$ and $\left[\frac{2\pi}{3},\pi\right]$. The midpoints of these intervals are $\frac{\pi}{6}$, $\frac{\pi}{2}$, and $\frac{5\pi}{6}$ (see picture at right). The heights of the three rectangles, therefore, are



- $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$
- $\sin\left(\frac{\pi}{2}\right) = 1$
- $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$.

Therefore the area of the rectangles, put together, is

$$\begin{aligned}\left(\frac{1}{2}+1+\frac{1}{2}\right)\Delta x &= \left(\frac{1}{2}+1+\frac{1}{2}\right)\cdot\frac{\pi}{3} \\ &= 2\cdot\frac{\pi}{3} = \boxed{\frac{2\pi}{3}}\end{aligned}$$

(b) Evaluate $\int_0^{\pi} \sin\theta \, d\theta$ exactly.

Using FTC-2, we get

$$\begin{aligned}\int_0^{\pi} \sin\theta \, d\theta &= -\cos\theta \Big|_0^{\pi} \\ &= (-\cos\pi) - (-\cos 0) = -(-1) + 1 = \boxed{2}\end{aligned}$$

(c) What is the error of the estimate you made in part (a)?

The error is $\frac{2\pi}{3} - 2 \approx 0.094$, a slight overestimate (π is a little more than 3, so $\frac{2\pi}{3}$ is a little more than 2). Looking at the graph and the rectangles, it seems plausible that our estimate would be a little too big, but not by much.

6. If $F(x) = \int_5^{\sin^2 x} (3t - 5)\, dt$,

(a) Evaluate $F'(x)$.

Using the chain rule, we get

$$F'(x) = (3\sin^2 x - 5) \cdot 2\sin x \cos x = \boxed{6\sin^3 x \cos x - 10 \sin x \cos x}$$

(b) Evaluate $F(x)$.

We have

$$\begin{aligned}F(x) &= \int_5^{\sin^2 x} (3t - 5)\, dt = \frac{3}{2}t^2 - 5t \Big|_5^{\sin^2 x} \\ &= \left(\frac{3}{2}(\sin^2 x)^2 - 5\sin^2 x\right) - \left(\frac{3}{2} \cdot 5^2 - 5 \cdot 5\right) \\ &= \boxed{\frac{3}{2}\sin^4 x - 5\sin^2 x - \frac{75}{2} + 25}\end{aligned}$$

(c) Show that the derivative of the function you obtained in (b) equals the function you obtained in (a).

Taking the derivative of the function $F(x)$ found in (a), we get

$$\begin{aligned}\frac{d}{dx}\left(\frac{3}{2}\sin^4 x - 5\sin^2 x - \frac{75}{2} + 25\right) &= \frac{3}{2} \cdot 4\sin^3 x \cos x - 10 \sin x \cos x \\ &= 6\sin^3 x \cos x - 10 \sin x \cos x,\end{aligned}$$

which is the answer we got in (a).

7. An object travels in a straight line with velocity function $v(t) = \frac{3}{t} - 4e^t$ feet per second. Determine the net change in position (in feet) over the time interval $2 \le t \le 5$.

The net change in position is $s(5)-s(2)$, where $s(t)$ is the position function. By F.T.C. we have

$$\begin{aligned}
s(5)-s(2) &= \int_2^5 \frac{3}{t} - 4e^t \, dt \\
&= 3\ln|t| - 4e^t \Big|_2^5 \\
&= \left(3\ln 5 - 4e^5\right) - \left(3\ln 2 - 4e^2\right) \\
&= 3(\ln 5 - \ln 2) - 4e^5 + 4e^2 \\
&= \boxed{3\ln\left(\tfrac{5}{2}\right) - 4e^2(e^3-1)}
\end{aligned}$$

8. Evaluate $\displaystyle\int x^3\cos(9x^4-2)\,dx$.

Let $u = 9x^4-2$. Then $du = 36x^3\,dx$, and we have

$$\begin{aligned}
\int x^3\cos(9x^4-2)\,dx &= \frac{1}{36}\int 36x^3\cos(9x^4-2)\,dx \\
&= \frac{1}{36}\int \cos(u)\,du \\
&= \frac{1}{36}\sin(u) + C \\
&= \boxed{\frac{1}{36}\sin(9x^4-2) + C}
\end{aligned}$$

9. Evaluate $\displaystyle\int \sin^3(x)\cos(x)\,dx$.

Let $u = \sin(x)$. Then $du = \cos(x)\,dx$, and we have

$$\begin{aligned}
\int \sin^3(x)\cos(x)\,dx &= \int u^3\,du \\
&= \frac{1}{4}u^4 + C \\
&= \boxed{\frac{1}{4}\sin^4(x) + C}
\end{aligned}$$

10. Evaluate $\displaystyle\int x\sin^3(x^2)\cos(x^2)\,dx$.

Hint. *You may find it helpful to use your answer to Work and Answer #9.*

You can either do this problem directly via the u-substitution $u = \sin(x^2)$, or you can make the substitution $u = x^2$ and then note that the integral turns into a constant multiple of #9. Here are the details of these two approaches:

Solution 1. Let $u = \sin(x^2)$. Then $du = 2x\cos(x^2)\ dx$, and we have

$$\begin{aligned}
\int x\sin^3(x^2)\cos(x^2)\ dx &= \frac{1}{2}\int 2x\sin^3(x^2)\cos(x^2)\ dx \\
&= \frac{1}{2}\int u^3\ du \\
&= \frac{1}{2}\cdot\frac{1}{4}u^4 + C \\
&= \boxed{\frac{1}{8}\sin^4(x^2) + C}
\end{aligned}$$

Solution 2. Let $u = x^2$. Then $du = 2x\ dx$, and we have

$$\begin{aligned}
\int x\sin^3(x^2)\cos(x^2)\ dx &= \frac{1}{2}\int 2x\sin^3(x^2)\cos(x^2)\ dx \\
&= \frac{1}{2}\int \sin^3(u)\cos(u)\ du \\
&= \frac{1}{2}\cdot\frac{1}{4}\sin^4(u) + C \qquad \text{(by \#9)} \\
&= \boxed{\frac{1}{8}\sin^4(x^2) + C}
\end{aligned}$$