Math 75B Selected Homework Solutions
Ch. 9 (E), §§2.6, 3.3 (S)

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§2.6 #8. Find $\frac{dy}{dx}$ by implicit differentiation: $1 + x = \sin(xy^2)$.

Taking the derivative of both sides, we get

$$1 = \cos(xy^2)\left(2xy\frac{dy}{dx} + y^2\right).$$

Now we solve for $\frac{dy}{dx}$:

$$\begin{aligned} 1 &= \cos(xy^2) \cdot 2xy\frac{dy}{dx} + \cos(xy^2) \cdot y^2 \\ 1 - y^2\cos(xy^2) &= 2xy\cos(xy^2)\frac{dy}{dx} \\ \frac{dy}{dx} &= \boxed{\frac{1 - y^2\cos(xy^2)}{2xy\cos(xy^2)}} \end{aligned}$$

§2.6 #16. If $g(x) + x\sin(g(x)) = x^2$, find $g'(0)$.

Differentiating implicitly, we get

$$g'(x) + x\cos(g(x)) \cdot g'(x) + \sin(g(x)) = 2x.$$

Since this equation is true for all x, it is certainly true for $x = 0$. Since the problem is asking for $g'(0)$, we plug in $x = 0$ to the equation and get

$$g'(0) + 0\cos(g(0)) \cdot g'(0) + \sin(g(0)) = 2 \cdot 0.$$

Simplifying, we get $g'(0) + \sin(g(0)) = 0$ which, when solved for $g'(0)$, gives

$$g'(0) = -\sin(g(0)).$$

All we need is to find $g(0)$. But the original equation is true for $x = 0$ (and every other x), so we also have

$$g(0) + 0\sin(g(0)) = 0^2$$

and therefore $g(0) = 0$. Thus $g'(0) = -\sin(g(0)) = -\sin(0) = \boxed{0}$

§3.3 #52. Use logarithmic differentiation to find the derivative of $y = \sqrt{x}^x$.

First we apply ln( ) to both sides: $\ln(y) = \ln\left(\sqrt{x}^x\right)$. Then we use logarithm laws to rewrite the right side:

$$\ln(y) = x\ln\left(\sqrt{x}\right) = \tfrac{1}{2}x\ln(x).$$

(Alternatively, we could have started this problem by noticing that $y = \left(x^{1/2}\right)^x = x^{x/2}$, and then applying ln() we get the same as above.)

Now we take the derivative implicitly on both sides and solve for y':

$$\begin{aligned}
\frac{y'}{y} &= \frac{x}{2}\cdot\frac{1}{x} + \frac{1}{2}\ln(x) \qquad \text{(notice that we used the product rule)} \\
&= 2 + \frac{\ln(x)}{2} \qquad \text{(simplifying)} \\
y' &= \left(2 + \frac{\ln(x)}{2}\right)y \\
&= \boxed{\left(2 + \frac{\ln(x)}{2}\right)\sqrt{x}^x}
\end{aligned}$$

§3.3 #58. Find y' if $x^y = y^x$.

First we need to apply ln() to get the variables out of the exponents. We have

$$\begin{aligned}
\ln(x^y) &= \ln(y^x) \\
y\ln(x) &= x\ln(y)
\end{aligned}$$

Now we take the derivative implicitly and solve for y':

$$\begin{aligned}
\frac{y}{x} + \ln(x)y' &= \frac{x}{y}y' + \ln(y) \\
\ln(x)y' - \frac{x}{y}y' &= \ln(y) - \frac{y}{x} \\
\left(\ln(x) - \frac{x}{y}\right)y' &= \ln(y) - \frac{y}{x} \\
y' &= \boxed{\frac{\ln(y) - \frac{y}{x}}{\ln(x) - \frac{x}{y}}}
\end{aligned}$$