Math 75B Selected Homework Solutions
§§14-C (E), 3.5 (S)

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§3.5 #16, 22. Find the derivative of the function and simplify.

§3.5 #16. $y = \sqrt{\tan^{-1}(x)}$

We have

$$\begin{aligned} y' &= \frac{1}{2}\left(\tan^{-1}(x)\right)^{-1/2} \cdot \frac{1}{1+x^2} \\ &= \boxed{\frac{1}{2(1+x^2)\sqrt{\tan^{-1}(x)}}} \end{aligned}$$

§3.5 #22. $h(t) = e^{\sec^{-1}(t)}$

From #14, we know (and have hopefully proved!) that $\frac{d}{dt}(\sec^{-1}(t)) = \frac{1}{t\sqrt{t^2-1}}$. Therefore we have

$$\begin{aligned} h'(t) &= e^{\sec^{-1}(t)} \cdot \frac{1}{t\sqrt{t^2-1}} \\ &= \boxed{\frac{e^{\sec^{-1}(t)}}{t\sqrt{t^2-1}}} \end{aligned}$$

§3.5 #42.*

(a) Sketch the graph of the function $f(x) = \sin(\sin^{-1}(x))$.

The domain of $\sin^{-1}(x)$ is $-1 \leq x \leq 1$. We know that for all such x, $\sin(\sin^{-1}(x)) = x$. So the graph of $f(x)$ is identical to that of $y = x$, except with the domain restricted to $-1 \leq x \leq 1$ (next page, left).

(b) Sketch the graph of the function $g(x) = \sin^{-1}(\sin(x))$ for all $x \in \mathbb{R}$.

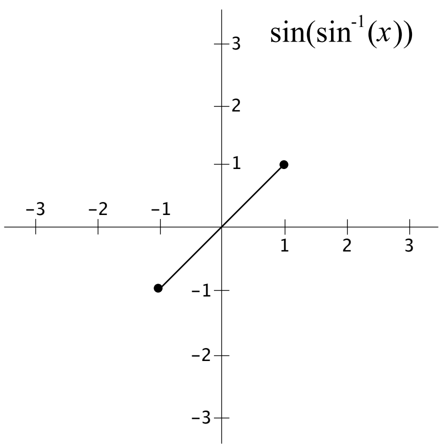
The domain of $\sin(x)$ is $\mathbb{R}$ (all real numbers). Therefore this is the domain of $g(x)$, so we will have to figure out what happens for *all* x.

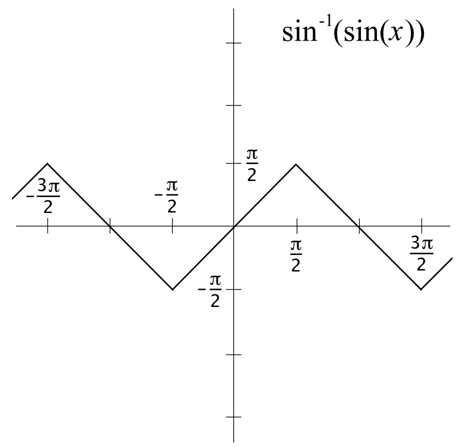
Quadrants IV and I. The range of $\sin^{-1}(x)$ is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. We know that for all x in this interval, $\sin^{-1}(\sin(x)) = x$. Since $g(x)$ is periodic with period 2π, the graph will repeat itself for all angles in quadrants IV in I (the preferred interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ consists of angles in quadrants IV and I).

Quadrant II. If x is an angle in quadrant II, say between $\frac{\pi}{2}$ and π, then $\sin(x)$ is positive. Therefore $\sin^{-1}(\sin(x))$ will also come out positive; in fact it will be the reference angle for x in the first quadrant, which is $\pi - x$. So the graph of $g(x)$ will look like the line $y = \pi - x$ for all x in the interval $\left[\frac{\pi}{2}, \pi\right]$. Since $g(x)$ is periodic, the graph will repeat itself for all angles in quadrant II.

Quadrant III. If x is an angle in quadrant III, say between π and $\frac{3\pi}{2}$, then $\sin(x)$ is negative. Therefore $\sin^{-1}(\sin(x))$ will also come out negative; in fact (recalling some heavy-duty trigonometry from your youth) it will be $\pi - x$, the angle in quadrant IV having the same reference angle as that of x. So the graph of $g(x)$ will look like the line $y = \pi - x$ for all x in the interval $\left[\pi, \frac{3\pi}{2}\right]$. Since $g(x)$ is periodic, the graph will repeat itself for all angles in quadrant III.

Whew! After all that, we get the graph shown (below right).





(c) Show that $g'(x) = \dfrac{\cos x}{|\cos x|}$.

We have

$$\begin{aligned}
g'(x) &= \frac{1}{\sqrt{1 - (\sin(x))^2}} \cdot \cos(x) \\
&= \frac{\cos x}{\sqrt{1 - \sin^2 x}} \\
&= \frac{\cos x}{\sqrt{\cos^2 x}} \qquad \text{(by the Pythagorean identity } \sin^2 x + \cos^2 x = 1) \\
&= \frac{\cos x}{|\cos x|}.
\end{aligned}$$

Another way to look at this is to consider the graph of $g(x)$. From the properties of absolute values, we know that

$$\frac{\cos x}{|\cos x|} = \begin{cases} \dfrac{\cos x}{\cos x} & \text{if } \cos x > 0 \\[2ex] \dfrac{\cos x}{-\cos x} & \text{if } \cos x < 0 \\[2ex] \text{undefined} & \text{if } \cos x = 0 \end{cases}$$

which simplifies to

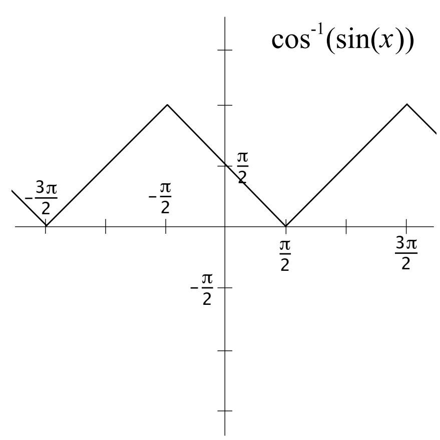
$$\begin{cases} 1 & \text{if } x \text{ is in quadrant IV or I (or in between)} \\ -1 & \text{if } x \text{ is in quadrant II or III (or in between)} \\ \text{undefined} & \text{if } x \text{ is an odd multiple of } \frac{\pi}{2} \end{cases} \qquad (1)$$

From the graph of $g(x)$ in part (b), we can see that the slope of the graph is 1 in quadrants IV and I (and in between, i.e. at multiples of 2π) and -1 in quadrants II and III (and in between, i.e. at odd multiples of π such as 3π, 5π, $-\pi$, etc.). You can also see that $g(x)$ is not differentiable at odd multiples of $\frac{\pi}{2}$ $\left(\frac{\pi}{2}, \frac{3\pi}{2}, \text{ etc.}\right)$ since there are sharp corners there. Therefore the derivative of $g(x)$ is exactly as given above in formula (1), so $g'(x) = \dfrac{\cos x}{|\cos x|}$.

(d) Sketch the graph of $h(x) = \cos^{-1}(\sin(x))$ for $x \in \mathbb{R}$ and find its derivative.

We can do an analysis similar to that in (b) to find the graph. It will also be a piecewise straight-line graph. For instance, if x is an acute angle, then we know that $\cos^{-1}(\sin x)$ is the complement of x, i.e. $\frac{\pi}{2} - x$. Also, the range of $h(x)$ is $[0, \pi]$, so we can expect the entire graph to be on or above the x-axis. We can get an idea of what the rest of the graph looks like by plotting key points in each quadrant:

x	$\sin(x)$	$\cos^{-1}(\sin(x))$
0	0	$\frac{\pi}{2}$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\pi}{3}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\pi}{6}$
$\frac{\pi}{2}$	1	0
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\pi}{6}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$\frac{\pi}{3}$
π	0	$\frac{\pi}{2}$
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{5\pi}{6}$
$\frac{3\pi}{2}$	-1	0
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{5\pi}{6}$



We get the graph shown (above right).
The derivative of $h(x)$ is

$$\begin{aligned}
h'(x) &= -\frac{1}{\sqrt{1-(\sin(x))^2}} \cdot \cos(x) \\
&= -\frac{\cos x}{\sqrt{1-\sin^2 x}} \\
&= -\frac{\cos x}{\sqrt{\cos^2 x}} \qquad \text{(by the Pythagorean identity } \sin^2 x + \cos^2 x = 1\text{)} \\
&= \boxed{-\frac{\cos x}{|\cos x|}}
\end{aligned}$$

Alternatively, from the graph we can see that the derivative of $h(x)$ is

$$h'(x) = \begin{cases} 1 & \text{if } x \text{ is in quadrant II or III (or in between)} \\ -1 & \text{if } x \text{ is in quadrant IV or I (or in between)} \\ \text{undefined} & \text{if } x \text{ is an odd multiple of } \frac{\pi}{2} \end{cases}$$

(which happens to be equal to $-\frac{\cos x}{|\cos x|}$).