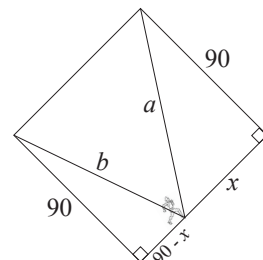**Math 75B Selected Homework Solutions**
§§Ch. 11 (E), 2.7 (S)

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§2.7 #16. A baseball diamond is a square with side 90 ft. A batter his the ball and runs toward first base with a speed of 24 ft./s.

(a) At what rate is his distance from second base decreasing when he is halfway to first base?

(b) At what rate is his distance from third base increasing when he is halfway to first base?

(a) Using the Pythagorean Theorem we have $x^2 + 90^2 = a^2$ (see the figure). Therefore the derivative is
$$2x\frac{dx}{dt} + 0 = 2a\frac{da}{dt}.$$

Plugging in $\frac{dx}{dt} = -24$ (the runner's distance x is *decreasing* as he runs to first base. We can also set up the problem by letting x be the distance of the runner from home plate and using $90 - x$ as the runner's distance from first base. In that case $\frac{dx}{dt}$ would be 24, not -24, since quantity x would be increasing). Also, we are supposed to find out what happens when the runner is halfway to first base, i.e. $x = 45$. Note that when $x = 45$, $a = \sqrt{45^2 + 90^2} = \sqrt{10125} = 45\sqrt{5}$. Plugging everything in and solving for $\frac{da}{dt}$ we get

$$2(45)(-24) = 2(45\sqrt{5})\frac{da}{dt}$$
$$\frac{da}{dt} = \frac{2(45)(-24)}{2(45\sqrt{5})} = -\frac{24}{\sqrt{5}}$$

Therefore the runner's distance from second base (which is *decreasing*, so we know $\frac{da}{dt}$ is supposed to come out negative!) is decreasing at a rate of $\boxed{\dfrac{24}{\sqrt{5}} \approx 10.7331 \text{ ft./s}}$

(b) At what rate is his distance from third base increasing when he is halfway to first base?

This is very similar to part (a), except we will use the other triangle in the picture (the one with the hypotenuse labeled b). We can either continue to use the x as in the picture (in which case our equation in this part will look like $(90 - x)^2 + 90^2 = b^2$), or we can rename things to make it easier: let $y = 90 - x$. Then our equation becomes

$$y^2 + 90^2 = b^2$$

whose derivative is

$$2y\frac{dy}{dt} + 0 = 2b\frac{db}{dt}.$$

Now we plug in $y = 45$, $\dfrac{dy}{dt} = 24$ (yes, *positive* 24, since the distance y is *increasing*), and $b = 45\sqrt{5}$ (when the runner is halfway to first base, the two hypotenuses are equal; see part (a) for the calculation), and we get

$$2(45)(24) = 2(45\sqrt{5})\frac{db}{dt}$$
$$\frac{db}{dt} = \frac{2(45)(24)}{2(45\sqrt{5})} = \frac{24}{\sqrt{5}}$$

Therefore the runner's distance from third base (which is *increasing*, so we know $\dfrac{db}{dt}$ is supposed to come out positive!) is increasing at a rate of $\boxed{\dfrac{24}{\sqrt{5}} \approx 10.7331 \text{ ft./s}}$

§2.7 #38.* The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

This is a very tricky problem and will require the Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

for a triangle with standard labeling (see picture at right).

Okay, so let's get started!

Using the Law of Cosines, we can set up an equation involving c and C (see picture at right) as follows:

$$c^2 = 8^2 + 4^2 - 2(4)(8)\cos(C)$$

which simplifies to

$$c^2 = 80 - 64\cos(C) \qquad\qquad (1)$$

You can see that c, the distance between the tips of the hands, is mentioned in the problem (our goal is to find $\frac{dc}{dt}$), but it may not be obvious that we need the angle C. But the rotation of the hands of a clock is the only movement that affects the distance c, and we can measure that rate of rotation.

Taking the derivative of the above equation, we have

$$2c\frac{dc}{dt} = 64\sin(C)\frac{dC}{dt}. \qquad\qquad (2)$$
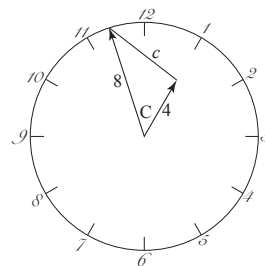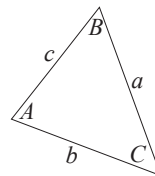
Now comes the fun part: at one o'clock, the angle C is $30°$, or $\dfrac{\pi}{6}$. Therefore $\sin(C) = \sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$. Using the equation (1), we can also solve for what c is at that moment:

$$c^2 = 80 - 64\cos\left(\frac{\pi}{6}\right)$$
$$= 80 - 64\frac{\sqrt{3}}{2} = 80 - 32\sqrt{3};$$

therefore $c = \sqrt{80 - 32\sqrt{3}}$.

Finally, there is $\dfrac{dC}{dt}$. We need to decide what units we are going to use for time. Let's use hours (though minutes works just as well). The minute hand goes an angle of 2π radians per hour. The hour hand goes an angle of $\dfrac{\pi}{6}$ radians per hour in the same direction. So the angle between them changes at a rate of $2\pi - \dfrac{\pi}{6} = \dfrac{11\pi}{6}$ radians per hour. At one o'clock, the angle between the hands is *decreasing* (look at or imagine a clock to visualize this), so we get

$$\frac{dC}{dt} = -\frac{11\pi}{6} \text{ radians per hour.}$$

Plugging all these numbers into the equation (2) and solving for $\dfrac{dc}{dt}$ we get

$$2\sqrt{80 - 32\sqrt{3}}\,\frac{dc}{dt} = 64 \cdot \frac{1}{2}\left(-\frac{11\pi}{6}\right)$$
$$\sqrt{80 - 32\sqrt{3}}\,\frac{dc}{dt} = -\frac{88\pi}{3}$$
$$\frac{dc}{dt} = -\frac{88\pi}{3\sqrt{80 - 32\sqrt{3}}} \approx -18.6 \text{ mm per hour}$$

Thus the distance between the tips of the hands is decreasing at a rate of approximately 18.6 mm per hour