Math 152 Practice for Midterm II
§§2.2-3.8

DISCLAIMER. This collection of practice problems is *not* guaranteed to be identical, in length, format or content, to the actual exam.

If I were you, I would:

- Know all of the definitions mentioned in class and in the sections of the book, and know examples relating to them.
- Know all of the theorems mentioned in class and in the sections of the book, and know examples relating to them.
- Go over all homework problems, even "redoing" them on WeBWorK in order to practice.
- Go over all quiz problems.
- Especially practice proving things.

You should also know how to do the following:

1. Find the determinant of a square matrix of any size
2. Find the adjoint of a square matrix
3. Use the adjoint formula to find the inverse of an invertible matrix
4. Use Gaussian/Gauss-Jordan elimination to solve a system of equations
5. Use Cramer's Rule to solve a system of equations
6. Determine whether or not a subset of a given vector space is a subspace of the vector space
7. Determine whether or not a given set of vectors forms a basis for a given vector space
8. Given a spanning set S of vectors for a vector space V, find a subset of S that is a basis of V.
9. Given a linearly independent set S of vectors in a vector space V, find a basis of V containing the vectors in S.
10. Determine the dimension of a given vector space
11. Find a basis for the null space of a matrix
12. Given a vector space V and an ordered basis S of V, find the coordinate vector of a given element $v \in V$ relative to S.
13. Know the definition of a *transition matrix* and how to compute it. Know the difference between $P_{T\to S}$ and $P_{S\to T}$.

Some sample questions:

1. **Multiple Choice.** *Circle the letter of the best answer.* The reduced row-echelon forms of the augmented matrices of four systems of equations are given below. How many solutions does each system have?

   (A) $\left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & -7 \end{array}\right]$

   (a) Unique solution

   (b) Infinitely many solutions

   (c) No solutions

   (d) None of the above

   (B) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 0 & 1 & -11 \end{array}\right]$

   (a) Unique solution

   (b) No solutions

   (c) Infinitely many solutions

   (d) None of the above

   (C) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 17 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

   (a) No solutions

   (b) Unique solution

   (c) Infinitely many solutions

   (d) None of the above

   (D) $\left[\begin{array}{cc|c} 1 & 0 & -15 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

   (a) Infinitely many solutions

   (b) No solutions

   (c) Unique solution

   (d) None of the above

2. (**W**)* Solve the system of equations

$$\begin{array}{rrrrl} w & +x & +2y & -z & = 2 \\ & +x & +y & +z & = 3 \\ -w & & -y & +2z & = 1 \end{array}$$

Express your solution(s), if one exists, as a vector.

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* **(W)** denotes a problem for which you must justify all steps to receive full credit. Most of the questions on the exam will be designated this way.

3. (**W**) Solve the system of equations

$$\begin{array}{rrrl} x_1 & -x_2 & & = 1 \\ 3x_1 & -5x_2 & +2x_3 & = 0 \\ & -x_2 & +x_3 & = -1 \end{array}$$

using **Cramer's Rule**. *Express your solution(s), if one exists, as a vector.*

4. (**W**) Suppose A, B, and C are $n \times n$ matrices and that $\det A = 3$, $\det B = -2$, and $\det C = 8$.

   (a) Prove that A is row equivalent to B.

   (b) Prove that $\det(A^2 B^{-1} C) = -36$.

5. (**W**) Prove that the set $W = \left\{ \begin{bmatrix} a \\ b \\ a-b \\ 2a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^4$.

6. (**W**) Find a basis for $\mathbb{R}^3$ that contains the vector $\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$.

7. (**W**) Find a subset of $S = \left\{ \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -8 \\ -7 \\ 4 \end{bmatrix} \right\}$ that is a basis for $\mathbb{R}^3$.

8. (**W**) Find a basis for the subspace W of P_3 spanned by the set

$$S = \left\{ 2t^3 - 1,\ -6t^3 - 2t^2 + 3,\ t^2,\ 4t^3 + t^2 - 2 \right\}.$$

   What is $\dim(W)$?

9. (**W**) For the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ 2 & 0 & 1 & -1 & 3 \\ 5 & -1 & 3 & 0 & 3 \\ 4 & -2 & 5 & 1 & 3 \\ 1 & 3 & -4 & -5 & 6 \end{bmatrix},$$

   (a) Find a basis for the null space of A.

   (b) Compute $\det A$.

10. (**W**) **Prove or Disprove.** *If the statement is true, prove it. Use definitions and theorems. If the statement is false, give a counterexample.*

   (a) Let A be an $m \times n$ matrix. Consider the set

   $$T = \left\{ \mathbf{v} \in \mathbb{R}^n \;\middle|\; A\mathbf{v} = \mathbf{0} \right\}.$$

   Then T a subspace of $\mathbb{R}^n$.

   (b) If a vector space V is spanned by $v_1, v_2, \ldots, v_n$ and if $W \neq \{0\}$ is a subspace of V, then W is contained in $\{v_1, v_2, \ldots, v_n\}$.

   (c) Every finite dimensional vector space contains only a finite number of vectors.

   (d) If a vector space V is spanned by $v_1, v_2, \ldots, v_n$ and $W \neq \{\mathbf{0}\}$ is a subspace of V, then W is spanned by a subset of $\{v_1, v_2, \ldots, v_n\}$.

   (e) Every real vector space contains an infinite number of vectors.

   (f) If $\{v_1, v_2, \ldots, v_n\}$ is a basis of a vector space V and $W \neq \{0\}$ is a subspace of V, then W contains at least one of the basis vectors $v_1, v_2, \ldots, v_n$.

11. (**W**) Prove the **Two Out of Three Rule** for bases of finite-dimensional vector spaces:

   Suppose S is a finite subset of an n-dimensional vector space V. If any two of the following hold, then so does the third (and hence S is a basis for V):

   - S has n vectors in it;
   - S is linearly independent;
   - S spans V.