**Math 76 Practice Problems for Midterm I - Solutions**
§§6.1-6.3

**DISCLAIMER.** This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

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**Multiple Choice.** *Circle the letter of the best answer.*

1. $\displaystyle\int_0^1 xe^x\,dx =$

   $\boxed{\text{(a)}}$ 1  (c) $e-1$

   (b) $e$  (d) 0

   Using integration by parts, we have

$$\begin{array}{ccc} u = x & & v = e^x \\ \downarrow & & \uparrow \\ du = dx & & dv = e^x\,dx \end{array}$$

   which gives

$$\begin{aligned}\int_0^1 xe^x\,dx &= xe^x\Big|_0^1 - \int_0^1 e^x\,dx \\ &= xe^x - e^x\Big|_0^1 \\ &= 1e^1 - e^1 - (0e^0 - e^0) \\ &= e - e - 0 + 1 = \boxed{1}\end{aligned}$$

2. $\displaystyle\int_{-\pi/4}^{\pi/4} \tan^2 x\,dx =$

   (a) $1+\dfrac{\pi}{2}$  $\boxed{\text{(c)}}$ $2-\dfrac{\pi}{2}$

   (b) $1-\dfrac{\pi}{4}$  (d) $2+\dfrac{\pi}{4}$

   Using the Pythagorean identity $\tan^2 x = \sec^2 x - 1$, we have

$$\begin{aligned}\int_{-\pi/4}^{\pi/4} \tan^2 x\,dx &= \int_{-\pi/4}^{\pi/4} (\sec^2 x - 1)\,dx \\ &= \tan x - x\Big|_{-\pi/4}^{\pi/4} \\ &= \left(\tan\left(\tfrac{\pi}{4}\right) - \tfrac{\pi}{4}\right) - \left(\tan\left(-\tfrac{\pi}{4}\right) + \tfrac{\pi}{4}\right) \\ &= 1 - \tfrac{\pi}{4} - (-1) - \tfrac{\pi}{4} = \boxed{2 - \tfrac{\pi}{2}}\end{aligned}$$

3. The partial fraction decomposition of $\dfrac{4x^3-2x+1}{(x^2+5)(x-3)^2}$ is

(a) $\dfrac{A}{x^2+5}+\dfrac{B}{x-3}+\dfrac{C}{(x-3)^2}$

**(b)** $\dfrac{Ax+B}{x^2+5}+\dfrac{C}{x-3}+\dfrac{D}{(x-3)^2}$

(c) $\dfrac{Ax+B}{x^2+5}+\dfrac{C}{(x-3)^2}$

(d) $\dfrac{Ax+B}{x^2+5}+\dfrac{C}{x-3}+\dfrac{Dx+E}{(x-3)^2}$

In the denominator we have one irreducible quadratic factor $x^2+5$, so we put a linear form in the numerator of that term. We also have a repeated linear factor $(x-3)^2$, so we put a constant form in the numerator of each power of $x-3$ up to the maximum $(x-3)^2$.

4. $\displaystyle\int \frac{2x-1}{(x+1)(x-2)}\,dx =$

**(a)** $\ln|x+1|+\ln|x-2|+C$

(b) $3\ln|x+1|-2\ln|x-2|+C$

(c) $\ln|x+1|-\ln|x-2|+C$

(d) $-\ln|x+1|+\ln|x-2|+C$

If $\dfrac{2x-1}{(x+1)(x-2)}=\dfrac{A}{x+1}+\dfrac{B}{x-2}$, then $A(x-2)+B(x+1)=2x-1$. Setting $x=2$ we see that $3B=3$, so $B=1$. Setting $x=-1$, we get $-3A=-3$, so $A=1$. Therefore

$$\begin{aligned}\int \frac{2x-1}{(x+1)(x-2)}\,dx &= \int \frac{1}{x+1}+\frac{1}{x-2}\,dx\\ &= \ln|x+1|+\ln|x-2|+C.\end{aligned}$$

5. $\displaystyle\int \frac{7}{x^2+6x+10}\,dx =$

(a) $\tan^{-1}\left(x+\frac{3}{7}\right)+C$

(b) $7\tan^{-1}(6x+10)+C$

(c) $\dfrac{7}{3}\tan^{-1}x+C$

**(d)** $7\tan^{-1}(x+3)+C$

By completing the square under the radical, we see that $x^2+6x+10=x^2+6x+9+10-9=(x+3)^2+1$, so we get

$$\int \frac{7}{x^2+6x+10}\,dx = 7\int \frac{1}{(x+3)^2+1}\,dx$$

Let $u=x+3$. Then $du=dx$, and we have

$$\begin{aligned}&= 7\int \frac{1}{u^2+1}\,du\\ &= 7\tan^{-1}(u)+C\\ &= 7\tan^{-1}(x+3)+C.\end{aligned}$$

**Fill-In.**

1. $\displaystyle\int \sec^3 x\, dx = \underline{\dfrac{1}{2}\sec x \tan x + \dfrac{1}{2}\ln|\sec x + \tan x| + C}$ .

   We did this problem in class. To recap: use parts and solve for the integral. Please see me if you would like to go over this important technique.

2. $\displaystyle\int \sin^3 x\, dx = \underline{-\cos x + \tfrac{1}{3}\cos^3 x + C}$ .

   We have

$$\begin{aligned}
\int \sin^3 x\, dx &= \int \sin^2 x \sin x\, dx \\
&= \int (1-\cos^2 x)\sin x\, dx \qquad \text{(Let } u = \cos x\text{. Then } du = -\sin x\, dx\text{)} \\
&= -\int (1-u^2)\, du \\
&= -\left(u - \tfrac{1}{3}u^3\right) + C \\
&= -\cos x + \tfrac{1}{3}\cos^3 x + C.
\end{aligned}$$

3. To evaluate the integral $\displaystyle\int \sqrt{5+x^2}\, dx$, it is best to use the trigonometric substitution

   $x = \underset{\text{(function of } \theta\text{)}}{\underline{\sqrt{5}\tan\theta}}$ .

   For the form $\sqrt{a^2+x^2}$ it is best to use the trigonometric substitution $x = a\tan\theta$, where it is understood that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. To see why, notice that the integral above becomes

$$\int \sqrt{5+5\tan^2\theta}\sqrt{5}\sec^2\theta\, d\theta = 5\int \sqrt{1+\tan^2\theta}\sec^2\theta\, d\theta$$

   (Notice the $1+\tan^2\theta$, which equals $\sec^2\theta$, under the square root; that is precisely the reason for the substitution we made)

$$= 5\int \sec\theta\sec^2\theta\, d\theta = \sqrt{5}\int \sec^3\theta\, d\theta,$$

   which now reduces to the integral in Fill-In #1.

4. If $\dfrac{x^2-3}{(x^2+1)(x-2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-2}$, then

   (a) $A = \underline{\dfrac{4}{5}}$

   (b) $B = \underline{\dfrac{8}{5}}$

   (c) $C = \underline{\dfrac{1}{5}}$

We have $(Ax+B)(x-2)+C(x^2+1) = x^2-3$. Setting $x=2$ we get $5C=1$, so $C=\frac{1}{5}$. Now $A+C=1$, so $A=\frac{4}{5}$. Finally $-2A+B=0$, so $B=\frac{8}{5}$.

**Work and Answer.** *You must show all relevant work to receive full credit.*

1. Evaluate the integral $\int x\sin 3x\ dx$.

   Using integration by parts, we have

$$\begin{array}{ccc} u = x & \qquad & v = -\frac{1}{3}\cos 3x \\ \downarrow & & \uparrow \\ du = dx & & dv = \sin 3x\ dx \end{array}$$

   which gives

$$\begin{aligned} \int x\sin 3x\ dx &= -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\ dx \\ &= \boxed{-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C} \end{aligned}$$

2. Evaluate the integral $\int x\sin^{-1}(x^2)\ dx$.

   This is similar to a problem we did in homework, but first we must make the $u$-substitution (actually we will use $t$ instead of $u$ because we will soon be using integration by parts, and we don't want to confuse the $u$'s): let $t=x^2$; then $dt = 2x\ dx$, and we get $\frac{1}{2}\int \sin^{-1}(t)\ dt$. Now using integration by parts, we have

$$\begin{array}{ccc} u = \sin^{-1}(t) & \qquad & v = t \\ \downarrow & & \uparrow \\ du = \frac{1}{\sqrt{1-t^2}}\ dt & & dv = dt \end{array}$$

   which gives

$$\frac{1}{2}\int \sin^{-1}(t)\ dt = \frac{1}{2}\left(t\sin^{-1}(t) - \int \frac{t}{\sqrt{1-t^2}}\right)\ dt$$

   (Now we use one final substitution for the remaining integral: $u = 1-t^2$. Then $du = -2t\ dt$.)

$$\begin{aligned} &= \frac{1}{2}\left(t\sin^{-1}(t) + \frac{1}{2}\int \frac{1}{\sqrt{u}}\ du\right) \\ &= \frac{1}{2}\left(t\sin^{-1}(t) + \frac{1}{2}\cdot 2u^{1/2}\right) + C \\ &= \frac{1}{2}\left(t\sin^{-1}(t) + \sqrt{1-t^2}\right) + C. \qquad \text{Finally we go back to } x\text{'s:} \\ &= \boxed{\frac{1}{2}x^2\sin^{-1}(x^2) + \frac{1}{2}\sqrt{1-x^4} + C} \end{aligned}$$

3. Evaluate the integral $\int e^x \sin x \, dx$.

Using integration by parts, we have

$$\begin{array}{ccc} u = e^x & \qquad & v = -\cos x \\ \downarrow & & \uparrow \\ du = e^x \, dx & & dv = \sin x \, dx \end{array}$$

which gives

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx.$$

This is similar to a problem that we did in class. Remember that we had to do parts twice and solve for the integral. So here's the second application of integration by parts:

$$\begin{array}{ccc} u = e^x & \qquad & v = \sin x \\ \downarrow & & \uparrow \\ du = e^x \, dx & & dv = \cos x \, dx \end{array}$$

We get $-e^x \cos x + \left(e^x \sin x - \int e^x \sin x\right)$. Now we are ready to solve for the integral; we have

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x,$$

so

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x + C.$$

Therefore

$$\int e^x \sin x \, dx = \boxed{-\frac{1}{2} e^x \cos x + \frac{1}{2} e^x \sin x + C}$$

4. Evaluate the integral $\int x^2 \ln(x^3) \, dx$.

Again, this is similar to problems we have done before. First we substitute $u = x^3$. Then $du = 3x^2 \, dx$, and we have $\frac{1}{3} \int \ln u \, du$. We showed in class that $\int \ln t \, du = t \ln t - t + C$ (you can rederive it by letting $u = \ln t$ and $dv = dt$ and using parts); therefore we get

$$\begin{aligned} \frac{1}{3} \int \ln u \, du &= \frac{1}{3}(u \ln u - u) + C \\ &= \boxed{\frac{1}{3}(x^3 \ln(x^3) - x^3) + C} \end{aligned}$$

If you are feeling clever, convince yourself that the above simplifies to $x^3 \left(\ln x - \frac{1}{3}\right) + C$ !

5. Evaluate the integral $\int \cos^2 x \sin^4 x \, dx$.

Since the powers of $\sin x$ and $\cos x$ are both even, we must use the half-angle and double-angle identities; we have

$$\begin{aligned}
\int \cos^2 x \sin^4 x \, dx &= \int (\cos x \sin x)^2 \sin^2 x \, dx \\
&= \int \left(\frac{1}{2} \sin 2x\right)^2 \left(\frac{1}{2}(1 - \cos 2x)\right) \, dx \\
&= \frac{1}{8} \int \sin^2 2x - \sin^2 2x \cos 2x \, dx \\
&= \frac{1}{8} \left[\int \left(\frac{1}{2}(1 - \cos 4x)\right) \, dx - \int \sin^2 2x \cos 2x \, dx\right]
\end{aligned}$$

(For the second integral, let $u = \sin 2x$ and proceed. I'll skip to the end of that. See me for details if you're not sure how I get there.)

$$= \boxed{\frac{1}{16}\left(x - \frac{1}{4} \sin 4x\right) - \frac{1}{48} \sin^3 2x + C}$$

6. Evaluate the integral $\int \tan^2 x \sec^4 x \, dx$.

This one is actually much nicer than Work and Answer #5; since the power of $\sec x$ is even we can use the Pythagorean identity. We have

$$\begin{aligned}
\int \tan^2 x \sec^4 x \, dx &= \int \tan^2 x \sec^2 x \sec^2 x \, dx \\
&= \int \tan^2 x (\tan^2 x + 1) \sec^2 x \, dx \qquad \text{(Let } u = \tan x\text{)} \\
&= \int u^2(u^2 + 1) \, du \\
&= \int u^4 + u^2 \, du \\
&= \boxed{\frac{1}{5} \tan^5 x + \frac{1}{3} \tan^3 x + C}
\end{aligned}$$

7. Evaluate the integral $\int \tan^3 x \sec^3 x \, dx$.

This one is also nice because the power of $\tan x$ is odd. We have

$$\begin{aligned}
\int \tan^3 x \sec^3 x \, dx &= \int \tan^2 x \sec^2 x \cdot \sec x \tan x \, dx \\
&= \int (\sec^2 x - 1) \sec^2 x \cdot \sec x \tan x \, dx \qquad \text{(Let } u = \sec x\text{)} \\
&= \int (u^2 - 1)u^2 \, du \\
&= \int u^4 - u^2 \, du \\
&= \boxed{\frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C}
\end{aligned}$$

8. Evaluate the integral $\displaystyle\int \tan^2 x \sec x \, dx$.

   This one could get ugly because the power of $\sec x$ is odd *and* the power of $\tan x$ is even. We have

$$\begin{aligned}
\int \tan^2 x \sec x \, dx &= \int (\sec^2 x - 1) \sec x \, dx \\
&= \int \sec^3 x - \sec x \, dx. \qquad \text{Using Fill-In \#1:} \\
&= \left(\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x|\right) - \ln|\sec x + \tan x| + C \\
&= \boxed{\frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + C}
\end{aligned}$$

9. Evaluate the integral $\displaystyle\int \cos 2x \sin 3x \, dx$.

   We have, using one of the product identities,

$$\begin{aligned}
\int \cos 2x \sin 3x \, dx &= \frac{1}{2}\int \sin x + \sin 5x \, dx \\
&= \frac{1}{2}\left(-\cos x - \frac{1}{5}\cos 5x\right) + C \\
&= \boxed{-\frac{1}{2}\left(\cos x + \frac{1}{5}\cos 5x\right) + C}
\end{aligned}$$

10. Evaluate the integral $\displaystyle\int \sqrt{4 - 9x^2} \, dx$.

    Here we must first see that $\displaystyle\int \sqrt{4-9x^2} \, dx = \int \sqrt{4-(3x)^2} \, dx = \frac{1}{3}\int \sqrt{4-u^2} \, du$ (where $u = 3x$), and then use the trigonometric substitution $u = 2\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

    [Alternatively, if you feel confident you can substitute directly from the $x$'s with $x = \frac{2}{3}\sin\theta$; then $dx = \frac{2}{3}\cos\theta \, d\theta$, and you'll get the same thing.]

    In either case we end up with

$$\begin{aligned}
\frac{1}{3}\int \sqrt{4 - 4\sin^2\theta} \cdot 2\cos\theta \, d\theta &= \frac{4}{3}\int \sqrt{1 - \sin^2\theta}\cos\theta \, d\theta \\
&= \frac{4}{3}\int \sqrt{\cos^2\theta}\cos\theta \, d\theta \\
&= \frac{4}{3}\int \cos\theta\cos\theta \, d\theta \qquad (\text{since } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}) \\
&= \frac{4}{3}\int \cos^2\theta \, d\theta \\
&= \frac{4}{3}\cdot\frac{1}{2}\int 1 + \cos 2\theta \, d\theta \\
&= \frac{2}{3}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C
\end{aligned}$$

$$= \frac{2}{3}\left(\theta + \sin\theta\cos\theta\right) + C$$

Now use $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ to get $\theta = \sin^{-1}\left(\frac{u}{2}\right) = \sin^{-1}\left(\frac{3x}{2}\right)$ and trigonometry similar to what was done in class to get $\cos\theta = \frac{\sqrt{4-u^2}}{2} = \frac{\sqrt{4-9x^2}}{2}$. We have

$$= \frac{2}{3}\left(\sin^{-1}\left(\frac{3x}{2}\right) + \frac{3x}{2}\cdot\frac{\sqrt{4-9x^2}}{2}\right) + C$$

$$= \boxed{\frac{2}{3}\left(\sin^{-1}\left(\frac{3x}{2}\right) + \frac{3x\sqrt{4-9x^2}}{4}\right) + C}$$

11. Evaluate the integral $\int \sqrt{4+9x^2}\,dx$.

Similar to above we use the substitution $x = \frac{2}{3}\tan\theta$; then $dx = \frac{2}{3}\sec^2\theta\,d\theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, and we get

$$\begin{aligned}
\frac{2}{3}\int \sqrt{4+4\sin^2\theta}\cdot\sec^2\theta\,d\theta &= \frac{4}{3}\int \sqrt{1+\tan^2\theta}\sec^2\theta\,d\theta \\
&= \frac{4}{3}\int \sqrt{\sec^2\theta}\sec^2\theta\,d\theta \\
&= \frac{4}{3}\int \sec^3\theta\,d\theta \qquad (\text{since } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}) \\
&= \frac{4}{3}\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right) + C \qquad (\text{Using Fill-In \#1 again})
\end{aligned}$$

Again using trigonometry we get $\tan\theta = \frac{3x}{2}$ and $\sec\theta = \frac{\sqrt{4+9x^2}}{2}$. We have

$$= \frac{2}{3}\left(\frac{\sqrt{4+9x^2}}{2}\cdot\frac{3x}{2} + \ln\left|\frac{\sqrt{4+9x^2}}{2} + \frac{3x}{2}\right|\right) + C$$

$$= \boxed{\frac{2}{3}\left(\frac{3x\sqrt{4+9x^2}}{4} + \ln\left|\frac{3x+\sqrt{4+9x^2}}{2}\right|\right) + C}$$

12. Evaluate the integral $\int \sqrt{9x^2-4}\,dx$.

Once more, with feeling! This time it's $x = \frac{2}{3}\sec\theta$ with $0 \le \theta < \frac{\pi}{2}$; then $dx = \frac{2}{3}\sec\theta\tan\theta\,d\theta$, and we get

$$\begin{aligned}
\frac{2}{3}\int \sqrt{4\sec^2\theta - 4}\cdot\sec\theta\tan\theta\,d\theta &= \frac{4}{3}\int \sqrt{\sec^2\theta - 1}\sec\theta\tan\theta\,d\theta \\
&= \frac{4}{3}\int \sqrt{\tan^2\theta}\sec\theta\tan\theta\,d\theta \\
&= \frac{4}{3}\int \tan^2\theta\sec\theta\,d\theta \qquad (\text{since } 0 \le \theta < \frac{\pi}{2}) \\
&= \frac{4}{3}\cdot\frac{1}{2}\left(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right) + C
\end{aligned}$$

(using Work and Answer #8). Again using trigonometry we get $\sec\theta = \dfrac{3x}{2}$ and $\tan\theta = \dfrac{\sqrt{9x^2-4}}{2}$. We have

$$= \frac{2}{3}\left(\frac{3x}{2}\cdot\frac{\sqrt{9x^2-4}}{2} - \ln\left|\frac{3x}{2}+\frac{\sqrt{9x^2-4}}{2}\right|\right) + C$$

$$= \boxed{\frac{2}{3}\left(\frac{3x\sqrt{9x^2-4}}{4} - \ln\left|\frac{3x+\sqrt{9x^2-4}}{2}\right|\right) + C}$$

13. Evaluate the integral $\displaystyle\int \frac{2x^2-5x+10}{(x-1)^3}\,dx$.

The integrand is a proper rational function with repeated linear factors. Hence we may split up the fraction as

$$\frac{2x^2-5x+10}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}.$$

Then by getting a common denominator for the right side of the above equation and setting the resulting numerator equal to $2x^2-5x+10$, we get

$$A(x-1)^2 + B(x-1) + C = 2x^2-5x+10$$
$$Ax^2 + (-2A+B)x + (A-B+C) = 2x^2-5x+10 \tag{1}$$

**Note.** By setting $x=1$ we can see that $C = 2-5+10 = 7$. However, it is difficult to use the "cover-up" method beyond this point since there is no other "easy" $x$-value to plug in to make terms drop out. Therefore we will proceed with the method of undetermined coefficients.

From (1) we have $A=2$, $-2A+B=-5$ (so $B=-1$), and $A-B+C=10$ (so $2-(-1)+C=10$, and $C=7$, as expected from the **Note** above). Therefore

$$\int \frac{2x^2-5x+10}{(x-1)^3}\,dx = \int\left(\frac{2}{x-1} - \frac{1}{(x-1)^2} + \frac{7}{(x-1)^3}\right)dx$$

Now use the substitution $u = x-1$ in each fraction to get

$$= \boxed{2\ln|x-1| + \frac{1}{x-1} - \frac{7}{2(x-1)^2} + C}$$

14. Evaluate the integral $\displaystyle\int \frac{x^2-9x-7}{(x+2)(x^2+1)}\,dx$.

The integrand is a proper rational function with a linear factor and an irreducible quadratic factor, no repeats. Hence we may split up the fraction as

$$\frac{x^2-9x-7}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}.$$

Then by getting a common denominator for the right side of the above equation and setting the resulting numerator equal to $x^2-9x-7$, we get

$$A(x^2+1) + (Bx+C)(x+2) = x^2-9x-7$$
$$(A+B)x^2 + (2B+C)x + (A+2C) = x^2-9x-7 \tag{2}$$

**Note.** By setting $x = -2$ we can see that $5A = 4 + 18 - 7 = 15$, so $A = 3$. However, it is difficult to use the "cover-up" method beyond this point since there is no other "easy" $x$-value to plug in to make terms drop out. Therefore we will proceed with the method of undetermined coefficients.

From (2) we have $A+B = 1$ (so $B = -2$ by the **Note** above), and $2B+C = -9$ (so $-4+C = -9$, and $C = -5$). We can also double-check that $A + 2C = -7$: sure enough, $A = 3$ and $C = -5$, so $A + 2C = 3 + 2(-5) = -7$. Therefore

$$\begin{aligned}\int \frac{x^2 - 9x - 7}{(x+2)(x^2+1)}\,dx &= \int \left(\frac{3}{x+2} - \frac{2x+5}{x^2+1}\right)\,dx \\ &= \int \frac{3}{x+2}\,dx - \left(\int \frac{2x}{x^2+1}\,dx + \int \frac{5}{x^2+1}\,dx\right)\end{aligned}$$

For the first integral, let $u = x + 2$. For the second integral, let $u = x^2 + 1$. For the third, refer to Multiple Choice #5:

$$= \boxed{3\ln|x+2| - \ln(x^2+1) - 5\tan^{-1} x + C}$$