Math 76 Practice Problems for Midterm III - Solutions
§§8.2-9.2

DISCLAIMER. This collection of practice problems is *not* guaranteed to be identical, in length or content, to the actual exam. You may expect to see problems on the test that are not exactly like problems you have seen before.

Multiple Choice. *Circle the letter of the best answer.*

1. $\displaystyle\sum_{n=1}^{\infty} 3\left(\tfrac{1}{2}\right)^n =$

   (a) 6

   $\boxed{\text{(b)}}$ 3

   (c) $\dfrac{3}{2}$

   (d) ∞ (diverges)

   This is a geometric series with $r = \dfrac{1}{2}$, so it converges. But take care! The sum is $\displaystyle\sum_{n=1}^{\infty} 3\left(\tfrac{1}{2}\right)^n = \frac{3}{1-\frac{1}{2}} - 3 = 6 - 3 = 3$ since the sum starts from $n = 1$, not $n = 0$.

2. The series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{n+2}}$

   (a) converges to $\dfrac{8}{9}$

   $\boxed{\text{(b)}}$ converges to $\dfrac{1}{9}$

   (c) converges to 3

   (d) converges to 9

   This is a geometric series. There are several ways to get it into a form that fits the formula. Here are two:

   Solution 1.

   We have $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{n+2}} = \sum_{n=3}^{\infty} \frac{2}{3^n}$.

   This is exactly in the form we want it, but there are three terms "missing." (The formula $\dfrac{a}{1-r}$ works when the series starts from $n = 0$, but this one starts at $n = 3$.) So we take $\dfrac{a}{1-r}$ and subtract off the terms corresponding to $n = 0$, $n = 1$, and $n = 2$. We get

   $$\frac{2}{1-\frac{1}{3}} - 2 - \frac{2}{3} - \frac{2}{9} = 3 - 2 - \frac{2}{3} - \frac{2}{9} = \frac{1}{9}.$$

   Solution 2.

   We have $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{n+2}} = \sum_{n=0}^{\infty} \frac{2}{3^{n+3}} = \sum_{n=0}^{\infty} \frac{2}{3^3}\left(\frac{1}{3}\right)^n = \frac{\frac{2}{27}}{1-\frac{1}{3}} = \frac{1}{9}.$

3. $\displaystyle\sum_{n=3}^{\infty}\left(\frac{2}{n}-\frac{2}{n+1}\right)=$

(a) 0

(b) $\frac{1}{6}$

$\boxed{\text{(c)}}$ $\frac{2}{3}$

(d) ∞ (diverges)

This is a telescoping series. The n-th partial sum is

$$s_n=\left(\frac{2}{3}-\frac{2}{4}\right)+\left(\frac{2}{4}-\frac{2}{5}\right)+\left(\frac{2}{5}-\frac{2}{6}\right)+\ldots+\left(\frac{2}{n}-\frac{2}{n+1}\right)=\frac{2}{3}-\frac{2}{n+1},$$

whose limit as $n\to\infty$ is $\frac{2}{3}$.

4. To determine whether or not the series $\displaystyle\sum_{n=2}^{\infty}\frac{5n^3}{1-2n+n^4}$ converges, the limit comparison test may be used with comparison series $\sum b_n=$

$\boxed{\text{(a)}}$ $\sum\frac{1}{n}$

(b) $\sum 5n^3$

(c) $\sum\frac{5}{n^4}$

(d) none; the limit comparison test cannot be used

The degree of the denominator of $a_n=\dfrac{5n^3}{1-2n+n^4}$ is one more than the degree of the numerator. So the best comparison term is $b_n=\dfrac{1}{n}$. To check, note that the limit of $\frac{a_n}{b_n}$ is finite and positive, since

$$\begin{aligned}\lim_{n\to\infty}\frac{\frac{5n^3}{1-2n+n^4}}{\frac{1}{n}}&=\lim_{n\to\infty}\frac{5n^3}{1-2n+n^4}\cdot\frac{n}{1}\\&=\lim_{n\to\infty}\frac{5n^4}{1-2n+n^4}=5.\end{aligned}$$

5. The series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n^2-4\sqrt{n}-1}$

(a) $\boxed{\text{converges absolutely (AC)}}$

(b) converges conditionally (CC)

(c) diverges

Since the biggest power on the bottom is n^2 and the biggest power on the top is $n^{1/2}$, the difference in the powers is greater than 1 ($2-\frac{1}{2}=\frac{3}{2}$). Thus the series converges absolutely (AC) by the limit comparison test, using $b_n=\dfrac{1}{n^{3/2}}$.

Here is a more detailed solution:

Try for AC: look at the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 - 4\sqrt{n} - 1}$. The terms a_n of this series are positive, at least from some point on. So we may use the limit comparison test. Let $b_n = \dfrac{1}{n^{3/2}}$. We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^{1/2}}{n^2 - 4n^{1/2} - 1} \cdot \frac{n^{3/2}}{1} \\
&= \lim_{n\to\infty} \frac{n^2}{n^2 - 4n^{1/2} - 1} = 1,
\end{aligned}$$

a finite positive limit. Therefore we are using the right b_n for the limit comparison test. Since $\sum b_n$ converges (it is a p-series with $p = \frac{3}{2}$), our series also converges. In other words, the original series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n^2 - 4\sqrt{n} - 1}$ converges *absolutely* (AC).

6. The series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n - 4\sqrt{n} - 1}$

   (a) converges absolutely (AC)

   (b) $\boxed{\text{converges conditionally (CC)}}$

   (c) diverges

   Here the difference in the powers is less than 1 ($1 - \frac{1}{2} = \frac{1}{2}$). Thus the series will not converge absolutely. However, it will still converge (conditionally), by the alternating series test: we have

   - Let $f(x) = \dfrac{\sqrt{x}}{x - 4\sqrt{x} - 1}$. Then

     $$f'(x) = \frac{\frac{1}{2\sqrt{x}}(x - 4\sqrt{x} - 1) - \sqrt{x}(1 - \frac{2}{\sqrt{x}})}{(x - 4\sqrt{x} - 1)^2} = -\frac{\frac{1}{2}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)}{(x - 4\sqrt{x} - 1)^2} < 0$$

     (I skipped a lot of algebra here; you can check my work). Therefore the terms are decreasing.

   - $\displaystyle\lim_{n\to\infty} \frac{\sqrt{n}}{n - 4\sqrt{n} - 1} = 0$ since the power on the bottom is bigger than the power on the top.

7. The series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{10^n}{7n!}$

   (a) $\boxed{\text{converges absolutely (AC)}}$

(b) converges conditionally (CC)

(c) diverges

Using the ratio test we get

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{10^{n+1}}{7(n+1)!}\cdot\frac{7n!}{10^n}\\ &= \lim_{n\to\infty}\frac{10}{n+1} = 0 < 1.\end{aligned}$$

Therefore the series is AC.

8. The series $\displaystyle\sum_{n=2}^{\infty}\left(\frac{2n^2+1}{n^2+5n-6}\right)^n$

(a) converges absolutely (AC)

(b) converges conditionally (CC)

(c) $\boxed{\text{diverges}}$

Note. There was a typo in this question; it originally said $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2n^2+1}{n^2+5n-6}\right)^n$, which unfortunately is not defined at $n = 1$. However, the procedure for determining what the series does *eventually* is the same:

Using the root test we get

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{|a_n|} &= \lim_{n\to\infty}\sqrt[n]{\left(\frac{2n^2+1}{n^2+5n-6}\right)^n}\\ &= \lim_{n\to\infty}\frac{2n^2+1}{n^2+5n-6} = 2 > 1.\end{aligned}$$

Therefore the series diverges.

9. The interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}(x-1)^n$ is

(a) $[0, 1]$

(b) $(0, 1)$

(c) $(0, 2]$

$\boxed{\text{(d)}}$ $[0, 2)$

Since the power series is centered at $x = 1$, we can see immediately that the answer must be either (c) or (d) (You can also check this using the ratio test).

Now we check the endpoints: at $x = 0$ we have $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ which converges by the alternating series test. At $x = 2$ we have $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges by the p-series test.

10. A power series representation for the function $f(x) = \dfrac{3}{4-x}$ is

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{3}{4} x^n$

(b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^{n+1} x^n$

$\boxed{\text{(c)}}$ $\displaystyle\sum_{n=0}^{\infty} \frac{3}{4^{n+1}} x^n$

(d) $\displaystyle\sum_{n=0}^{\infty} 3(4-x)^n$

We have

$$\begin{aligned}\frac{3}{4-x} &= \frac{3}{4\left(1-\frac{1}{4}x\right)} = \frac{\frac{3}{4}}{1-\frac{1}{4}x} \\ &= \sum_{n=0}^{\infty} \frac{3}{4}\left(\frac{1}{4}x\right)^n = \sum_{n=0}^{\infty} \frac{3}{4^{n+1}} x^n.\end{aligned}$$

11. The Maclaurin series for the function $f(x) = x^3 \cos(4x^2)$ is

$\boxed{\text{(a)}}$ $\displaystyle\sum_{n=0}^{\infty} \frac{(-16)^n}{(2n)!} x^{4n+3}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n+3}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (4x^2)^n$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n^2+3}$

Recall that the Maclaurin series for $\cos(x)$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, for all x. Therefore for $\cos(4x^2)$ it is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (4x^2)^{2n}$. Finally, for $x^3\cos(4x^2)$ it is

$$x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (4x^2)^{2n} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{4n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-16)^n}{(2n)!} x^{4n+3}.$$

12. The equation of the line tangent to the curve $\begin{array}{c} x = e^{\sqrt{t}} \\ y = t - \ln t^2 \end{array}$ at the point corresponding to $t = 4$ is

(a) $y = \dfrac{2}{e^2} x + 4 - \ln 16$

(b) $y = \dfrac{1}{2} x + 4 - \ln 16 - \dfrac{1}{2} e^2$

(c) $y = \dfrac{e^2}{4} x + 4 - \ln 16 - \dfrac{1}{4} e^4$

$\boxed{\text{(d)}}$ $y = \dfrac{2}{e^2} x + 2 - \ln 16$

The slope is $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ evaluated at $t = 4$. Since $\dfrac{dy}{dt} = 1 - \dfrac{2}{t} = \dfrac{t-2}{t}$ and $\dfrac{dx}{dt} = \dfrac{1}{2} t^{-\frac{1}{2}} e^{\sqrt{t}} = \dfrac{e^{\sqrt{t}}}{2\sqrt{t}}$, we have $\dfrac{dy}{dx} = \dfrac{t-2}{t} \cdot \dfrac{2\sqrt{t}}{e^{\sqrt{t}}} = \dfrac{2(t-2)\sqrt{t}}{te^{\sqrt{t}}}$, which evaluated at $t = 4$ is $\dfrac{2}{e^2}$. Now the point on the curve corresponding to $t = 4$ (the point of tangency) is $(e^2, 4 - \ln 16)$, so $4 - \ln 16 = \frac{2}{e^2}e^2 + b = 2 + b$. Therefore $b = 2 - \ln 16$, and the equation of the line is $y = \frac{2}{e^2}x + 2 - \ln 16$.

13. The length of the curve $\begin{array}{c} x = \cos^2 t \\ y = \cos t \end{array}$ is

(a) $\int_0^{2\pi} \sqrt{\sin^2 2t + \sin^2 t}\, dt$ (c) $\int_0^{2\pi} \sqrt{1 + 14\sec^2 t}\, dt$

$\boxed{\text{(b)}}$ $\int_0^{\pi} \sqrt{\sin^2 2t + \sin^2 t}\, dt$ (d) $\int_0^{\pi} \sqrt{1 + 14\sec^2 t}\, dt$

Since the endpoints are not given it is up to us to find how long the curve is. After eliminating the parameter we have $x = y^2$, so it looks like a sideways parabola. But y can only have values between -1 and 1 since it is equal to the cosine of something. When $t = 0$, $y = 1$; and when $t = \pi$, $y = -1$. There are no angles between 0 and π that will make $y = \pm 1$, so the curve is traced once from $t = 0$ to $t = \pi$.

We have $x' = 2\cos t \sin t = \sin 2t$, so $(x')^2 = \sin^2 2t$. We also have $y' = -\sin t$, so $(y')^2 = \sin^2 t$. Therefore the arc length is $\int_0^{\pi} \sqrt{\sin^2 2t + \sin^2 t}\, dt$.

Fill-In.

1. $\sum_{n=1}^{\infty} \frac{5^n}{n^2+1}(x+3)^{n-1}$ is a power series centered at $\underline{\quad -3 \quad}$.

A power series of the form $\sum_{n=?}^{\infty} c_n(x-a)^n$ is centered at a. Here we have $a = -3$.

2. The radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} x^n$ is $\underline{\quad 1 \quad}$.

Using the ratio test, we have

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n} \right| &= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} |x| \\ &= |x| \overset{\text{set}}{<} 1. \end{aligned}$$

Therefore the radius of convergence is $R = 1$.

3. *Circle the best answer. On the line, indicate one valid test that can be applied to get your answer. You may choose from the following list:*

- divergence test
- p-series test
- geometric series test
- integral test
- direct comparison test
- limit comparison test
- alternating series test
- ratio test
- root test

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^3}$ ($\boxed{\textbf{converges}}$ | **diverges**)

Test: <u>p-series test, integral test, limit comparison test</u>

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3-1}{3n^2}$ (**converges** | $\boxed{\textbf{diverges}}$) Test: <u>divergence test</u>

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3\sqrt{n}}{n^2-3n+1}$ ($\boxed{\textbf{converges}}$ | **diverges**)

Test: <u>limit comparison test, integral test</u>

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\tan^{-1}(n)}$ (**converges** | $\boxed{\textbf{diverges}}$) Test: <u>divergence test</u>

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(5n)!}$ ($\boxed{\textbf{converges}}$ | **diverges**) Test: <u>ratio test</u>

Graph. *More accuracy = more points!* Let C be the curve $\begin{array}{c} x = \cos t \\ y = \sin t \cos t \end{array}$.

(a) Eliminate the parameter to find a Cartesian equation of C.

$y^2 = \sin^2 t \cos^2 t = (1-\cos^2 t)\cos^2 t = (1-x^2)x^2$. So we have $\boxed{y^2 = (1-x^2)x^2 \text{ or } y = \pm x\sqrt{1-x^2}}$

(b) Find the point(s) on the curve where the tangent is vertical.

Since the curve is periodic with period 2π, we only need to consider t-values between 0 and 2π.

$\dfrac{dy}{dx} = y'x' = -\sin^2 t + \cos^2 t - \sin t = -\cos 2t \sin t$.

The tangents will be vertical at those t-values for which $\dfrac{dy}{dx}$ is undefined (the denominator $= 0$). So set $\sin t = 0$. The solutions are 0 and π. The points corresponding to these t-values are $\boxed{(1,0) \text{ and } (-1,0)}$

(c) Find the point(s) on the curve where the tangent is horizontal.

The tangents will be horizontal at those t-values for which $\dfrac{dy}{dx}$ is 0 (the numerator $= 0$).

Again, we only need to look for solutions on the interval $[0, 2\pi]$. So set $\cos 2t = 0$. The solutions are $t = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}$, and $\dfrac{7\pi}{4}$. The points corresponding to these t-values are

$$\boxed{\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{1}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{1}{2}\right), \text{ and } \left(\frac{\sqrt{2}}{2}, -\frac{1}{2}\right)}$$

(d) Find equation(s) of the tangent(s) to C at the point $(0,0)$.

Set $x = 0$ and $y = 0$:

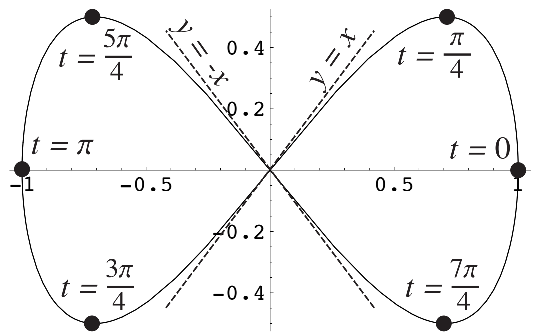
$0 = \cos t \Rightarrow t = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$ are the only solutions in the interval $[0, 2\pi]$. Both of these t-values satisfy $0 = \sin t \cos t$, so there are two t-values corresponding to the point $(0,0)$.

The slopes are as follows:

- $t = \dfrac{\pi}{2}$: $y'\left(\dfrac{\pi}{2}\right) = -\dfrac{\cos 2(\frac{\pi}{2})}{\sin \frac{\pi}{2}} = 1$
- $t = \dfrac{3\pi}{2}$: $y'\left(\dfrac{3\pi}{2}\right) = -\dfrac{\cos 2(\frac{3\pi}{2})}{\sin \frac{3\pi}{2}} = -1$

The y-intercept is $(0,0)$ in both cases, so the equations are $y = x$ and $y = -x$.

(e) Sketch a graph of C, labeling the features found in parts (b)-(d).



Work and Answer. *You must show all relevant work to receive full credit.*

1. Find the sum of the series $\displaystyle\sum_{n=-1}^{\infty} \frac{2 \cdot 3^n}{4^{n-1}}$.

   This is a geometric series after some manipulation. Note that

$$\sum_{n=-1}^{\infty} \frac{2 \cdot 3^n}{4^{n-1}} = \sum_{n=-2}^{\infty} \frac{2 \cdot 3^{n+1}}{4^n} = \sum_{n=-2}^{\infty} \frac{6 \cdot 3^n}{4^n} = \sum_{n=-2}^{\infty} 6 \cdot \left(\frac{3}{4}\right)^n$$

$$= \frac{6}{1 - \frac{3}{4}} + 6 \cdot \left(\frac{3}{4}\right)^{-1} + 6 \cdot \left(\frac{3}{4}\right)^{-2}$$

$$= 24 + 6\left(\frac{4}{3}\right) + 6\left(\frac{16}{9}\right) = \boxed{\frac{128}{3}}$$

2. Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \frac{n+1}{n^3-n}$.

This is a telescoping series after some manipulation. Note that

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{n+1}{n^3-n} &= \sum_{n=2}^{\infty} \frac{n+1}{n(n+1)(n-1)} \\ &= \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right)\end{aligned}$$

(using partial fractions). Therefore

$$\begin{aligned}s_n &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 1 - \frac{1}{n}\end{aligned}$$

which approaches 1 as $n \to \infty$. Therefore the sum of the series is $\boxed{1}$

3. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n \sin n}{n!}$ is absolutely convergent (AC), conditionally convergent (CC), or divergent.

Try for AC: we want to check whether or not $\displaystyle\sum_{n=1}^{\infty} \left|\frac{3^n \sin n}{n!}\right|$ converges or not. We have

$$\left|\frac{3^n \sin n}{n!}\right| = \frac{3^n |\sin n|}{n!} \leq \frac{3^n}{n!}$$

since $|\sin n| \leq 1$ for all n. We are attempting to use the direct comparison test — however, we need another test to determine whether or not $\sum \frac{3^n}{n!}$ converges. You can check using the ratio test that $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$ does converge. Therefore $\displaystyle\sum_{n=1}^{\infty} \left|\frac{3^n \sin n}{n!}\right|$ also converges, and hence the original series $\boxed{\textbf{converges absolutely (AC)}}$

4. (a) Find a power series representation for the function $f(x) = \ln(2+3x)$.

There are two ways to do this problem. One is to use §11.10 and find, say, the Maclaurin series for $f(x)$. This is quite difficult, however. Here's the way I recommend:

Use §11.9 and recognize that $f'(x) = \dfrac{3}{2+3x} = \dfrac{\frac{3}{2}}{1 - \left(-\frac{3}{2}x\right)}$, which is the sum

$$\sum_{n=0}^{\infty} \frac{3}{2}\left(-\frac{3}{2}x\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{n+1}}{2^{n+1}} x^n,$$

and therefore $f(x)$ is equal to an antiderivative of this sum: we get

$$\ln(2+3x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{n+1}}{(n+1)2^{n+1}} x^{n+1} + C = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 3^n}{n \cdot 2^n} x^n + C.$$

Plugging in $x = 0$ we see that $\ln(2+3(0)) = \ln 2 = C$. Therefore we have

$$\ln(2+3x) = \boxed{\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 3^n}{n \cdot 2^n} x^n}$$

(b) Find the interval of convergence.

Using the ratio test, we have

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{3^{n+1} x^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n \cdot 2^n}{3^n x^n} \right| \\ &= \lim_{n\to\infty} \frac{3n}{2(n+1)} |x| = \frac{3}{2}|x| \overset{\text{set}}{<} 1\end{aligned}$$

So $|x| < \frac{2}{3}$. It remains to check the endpoints.

When $x = \frac{2}{3}$ we have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 3^n}{n \cdot 2^n} \left(\frac{2}{3}\right)^n$ (the $(\ln 2)$ at the beginning won't affect whether the series converges or not), which equals $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$. This **converges** by the alternating series test.

When $x = -\frac{2}{3}$ we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 3^n}{n \cdot 2^n} \left(-\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n} = \sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n},$$

which **diverges** since it is a p-series with $p = 1$.

Therefore the interval of convergence is $\boxed{\left(-\frac{2}{3}, \frac{2}{3}\right]}$

5. (a) Write the Taylor series for the function $f(x) = \sqrt{x}$ centered at 1.

We have

$$
\begin{aligned}
f(x) &= \sqrt{x} & f(1) &= 1 \\
f'(x) &= \frac{1}{2\sqrt{x}} & f'(1) &= \frac{1}{2} \\
f''(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2} & f''(1) &= \frac{1}{2}\left(-\frac{1}{2}\right) \\
f'''(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2} & f'''(1) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \\
f^{(4)}(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2} & f^{(4)}(1) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) \quad \ldots
\end{aligned}
$$

We can see now the pattern that we get. We have

$$
\begin{aligned}
f^{(5)}(1) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right) = \frac{1\cdot 3\cdot 5\cdot 7}{2^5} \\
f^{(6)}(1) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\left(-\frac{9}{2}\right) = -\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2^6} \\
\ldots \quad f^{(n)}(1) &= (-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}.
\end{aligned}
$$

By Taylor's Theorem the coefficients of the Taylor series for $f(x)$ are $c_n = \dfrac{f^{(n)}(1)}{n!}$. For $n \geq 2$ this follows the pattern as above, and we have $c_n = (-1)^{n-1}\dfrac{1\cdot 3\cdot 5\cdots(2n-3)}{n!\ 2^n}$. The first two terms (for $n = 0$ and $n = 1$) do not follow this pattern, so we just write them out separately; we get

$$
\boxed{\sqrt{x} = 1 + \frac{1}{2}(x-1) + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!\ 2^n}(x-1)^n}
$$

(b) Find the radius of convergence.

Using the ratio test we have

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{1\cdot 3\cdot 5\cdots(2n-1)\cdot(x-1)^{n+1}}{(n+1)!\ 2^{n+1}} \cdot \frac{n!\ 2^n}{1\cdot 3\cdot 5\cdots(2n-3)\cdot(x-1)^n}\right| \\
&= \lim_{n\to\infty}\frac{2n-1}{2(n+1)}|x-1| \\
&= |x-1| \overset{\text{set}}{<} 1.
\end{aligned}
$$

Therefore the radius of convergence is $\boxed{R = 1}$

(c) Estimate $\sqrt{1.4}$ using the first three terms of the Taylor series.

$\sqrt{1.4} \approx 1 + \dfrac{1}{2}(0.4) - \dfrac{1}{2!\cdot 4}(0.4)^2 = 1 + 0.2 - 0.02 = \boxed{1.18}$

(For comparison, a calculator gives $\sqrt{1.4} \approx 1.1832$.)

6. Estimate $\int_0^1 e^{x^2}\,dx$ using the first two terms of the Maclaurin series expansion.

First we have that the Maclaurin series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Since the radius of convergence is infinite we can substitute in x^2 for x to get the Maclaurin series for e^{x^2}, which is $\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$. Integrating term by term we get

$$\begin{aligned}
\int_0^1 e^{x^2}\,dx &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!\,(2n+1)} \Bigg|_0^1 \\
&= \left(\sum_{n=0}^{\infty} \frac{1^{2n+1}}{n!\,(2n+1)}\right) - \left(\sum_{n=0}^{\infty} \frac{0^{2n+1}}{n!\,(2n+1)}\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!\,(2n+1)}.
\end{aligned}$$

Now just evaluate the first 2 terms to get the approximation:

$$\approx \frac{1}{1} + \frac{1}{3} = \boxed{\frac{4}{3}}$$

7. Estimate $\int_0^1 \sin x^2\,dx$ using the first two terms of the Maclaurin series expansion.

This is similar to the previous problem.

First we have that the Maclaurin series for $\sin x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Since the radius of convergence is infinite we can substitute in x^2 for x to get the Maclaurin series for $\sin(x^2)$, which is $\sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$. Integrating term by term we get

$$\begin{aligned}
\int_0^1 \sin x^2\,dx &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)\cdot(2n+1)!} \Bigg|_0^1 \\
&= \left(\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 1^{4n+3}}{(4n+3)\cdot(2n+1)!}\right) - \left(\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 0^{4n+3}}{(4n+3)\cdot(2n+1)!}\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)\cdot(2n+1)!}.
\end{aligned}$$

Now just evaluate the first 2 terms to get the approximation:

$$\approx \frac{1}{3\cdot 1!} - \frac{1}{7\cdot 3!} = \boxed{\frac{13}{42}}$$

8. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \, \pi^{2n+1}}{(2n+1)!}$.

This is the Maclaurin series for $\sin x$ with $x = \pi$. Since the Maclaurin series converges to $\sin x$ for all x, the series above converges to $\sin \pi = \boxed{0}$

9. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \, \pi^{2n}}{4^n \, (2n)!}$.

This looks similar to the Maclaurin series for $\cos x$ with something plugged in for x. But there is a 4^n in the denominator, so we must evaluate carefully to make sure. We can rewrite the above as

$$\sum_{n=0}^{\infty} \frac{(-1)^n \, \left(\frac{\pi}{2}\right)^{2n}}{(2n)!}$$

which now looks exactly like $\cos x$ with $x = \frac{\pi}{2}$. Since the Maclaurin series converges to $\cos x$ for all x, the series above converges to $\cos\left(\frac{\pi}{2}\right) = \boxed{0}$

10. Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \frac{5^n}{n!}$.

This looks like the Maclaurin series for e^x (which converges to e^x for all x) with $x = 5$, except that our series starts at $n = 2$ rather than $n = 0$. So to get the sum of our series we will need to subtract off the "missing" $n = 0$ and $n = 1$ terms from the Maclaurin series. We have

$$\sum_{n=2}^{\infty} \frac{5^n}{n!} = \sum_{n=0}^{\infty} \frac{5^n}{n!} - \frac{5^0}{0!} - \frac{5^1}{1!} = e^5 - 1 - 5 = \boxed{e^5 - 6}$$

11. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(\sqrt{3})^{2n+1} \, (2n+1)}$.

Recall that

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{2n+1}$$

with radius of convergence $R = 1$. So our series looks like it could be $\tan^{-1}(x)$ with something plugged in, as long as what we are plugging in is between -1 and 1. We can rewrite our series as

$$\sum_{n=0}^{\infty} \frac{(-1)^n \, \left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1};$$

since $-1 < \frac{1}{\sqrt{3}} < 1$ we see that the series above converges to $\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \boxed{\dfrac{\pi}{6}}$

12. Set up, **but do not evaluate,** an integral for the length of the curve $\begin{matrix} x = \frac{1}{3}t^3 \\ y = \cos t \end{matrix}$ with $0 \le t \le \frac{\pi}{2}$.

We have $(x')^2 = t^4$ and $(y')^2 = (-\sin t)^2 = \sin^2 t$. So the arc length is

$$\boxed{L = \int_0^{\pi/2} \sqrt{t^4 + \sin^2 t}\ dt}$$

13. For the curve $\begin{matrix} x = 1 + \tan t \\ y = \cos 2t \end{matrix}$, find $\frac{dy}{dx}$.

$\frac{dy}{dt} = -2\sin 2t$ and $\frac{dx}{dt} = \sec^2 t$, so $\frac{dy}{dx} = \frac{-2\sin 2t}{\sec^2 t} = \boxed{-4\sin t \cos^3 t}$