More explicitly: if a is a value of x such that $Q(a) = 0$, then $Q(x) \neq 0$ for all x sufficiently close to a. Thus,

$$F(a) = \lim_{x \to a} F(x) \text{ [by continuity of } F] = \lim_{x \to a} G(x) \quad [\text{whenever } Q(x) \neq 0]$$
$$= G(a) \quad [\text{by continuity of } G]$$

68. Let $f(x) = ax^2 + bx + c$. We calculate the partial fraction decomposition of $\dfrac{f(x)}{x^2(x+1)^3}$. Since $f(0) = 1$, we must have $c = 1$, so $\dfrac{f(x)}{x^2(x+1)^3} = \dfrac{ax^2 + bx + 1}{x^2(x+1)^3} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2} + \dfrac{E}{(x+1)^3}$. Now in order for the integral not to contain any logarithms (that is, in order for it to be a rational function), we must have $A = C = 0$, so $ax^2 + bx + 1 = B(x+1)^3 + Dx^2(x+1) + Ex^2$. Equating constant terms gives $B = 1$, then equating coefficients of x gives $3B = b \Rightarrow b = 3$. This is the quantity we are looking for, since $f'(0) = b$.

7.5 Strategy for Integration

1. $\displaystyle\int \frac{\sin x + \sec x}{\tan x}\,dx = \int \left(\frac{\sin x}{\tan x} + \frac{\sec x}{\tan x}\right) dx = \int (\cos x + \csc x)\,dx = \sin x + \ln|\csc x - \cot x| + C$

2. $\displaystyle\int \tan^3\theta\,d\theta = \int (\sec^2\theta - 1)\tan\theta\,d\theta = \int \tan\theta\sec^2\theta\,d\theta - \int \frac{\sin\theta}{\cos\theta}\,d\theta$

$$= \int u\,du + \int \frac{dv}{v} \quad \begin{bmatrix} u = \tan\theta, & v = \cos\theta, \\ du = \sec^2\theta\,d\theta & dv = -\sin\theta\,d\theta \end{bmatrix}$$
$$= \tfrac{1}{2}u^2 + \ln|v| + C = \tfrac{1}{2}\tan^2\theta + \ln|\cos\theta| + C$$

3. $\displaystyle\int_0^2 \frac{2t}{(t-3)^2}\,dt = \int_{-3}^{-1} \frac{2(u+3)}{u^2}\,du \quad [u = t-3,\ du = dt] = \int_{-3}^{-1}\left(\frac{2}{u} + \frac{6}{u^2}\right) du = \left[2\ln|u| - \frac{6}{u}\right]_{-3}^{-1}$

$$= (2\ln 1 + 6) - (2\ln 3 + 2) = 4 - 2\ln 3 \text{ or } 4 - \ln 9$$

4. Let $u = x^2$. Then $du = 2x\,dx \Rightarrow \displaystyle\int \frac{x\,dx}{\sqrt{3 - x^4}} = \frac{1}{2}\int \frac{du}{\sqrt{3 - u^2}} = \frac{1}{2}\sin^{-1}\frac{u}{\sqrt{3}} + C = \frac{1}{2}\sin^{-1}\frac{x^2}{\sqrt{3}} + C.$

5. Let $u = \arctan y$. Then $du = \dfrac{dy}{1+y^2} \Rightarrow \displaystyle\int_{-1}^{1} \frac{e^{\arctan y}}{1+y^2}\,dy = \int_{-\pi/4}^{\pi/4} e^u\,du = [e^u]_{-\pi/4}^{\pi/4} = e^{\pi/4} - e^{-\pi/4}.$

6. $\displaystyle\int x\csc x\cot x\,dx \quad \begin{bmatrix} u = x, & dv = \csc x\cot x\,dx, \\ du = dx & v = -\csc x \end{bmatrix} = -x\csc x - \int (-\csc x)\,dx$

$$= -x\csc x + \ln|\csc x - \cot x| + C$$

7. $\displaystyle\int_1^3 r^4\ln r\,dr \quad \begin{bmatrix} u = \ln r, & dv = r^4\,dr, \\ du = \dfrac{dr}{r} & v = \dfrac{1}{5}r^5 \end{bmatrix} = \left[\frac{1}{5}r^5\ln r\right]_1^3 - \int_1^3 \frac{1}{5}r^4\,dr = \frac{243}{5}\ln 3 - 0 - \left[\frac{1}{25}r^5\right]_1^3$

$$= \tfrac{243}{5}\ln 3 - \left(\tfrac{243}{25} - \tfrac{1}{25}\right) = \tfrac{243}{5}\ln 3 - \tfrac{242}{25}$$

8. $\dfrac{x-1}{x^2-4x-5} = \dfrac{x-1}{(x-5)(x+1)} = \dfrac{A}{x-5} + \dfrac{B}{x+1} \quad \Rightarrow \quad x - 1 = A(x+1) + B(x-5)$. Setting $x = -1$ gives $-2 = -6B$, so $B = \frac{1}{3}$. Setting $x = 5$ gives $4 = 6A$, so $A = \frac{2}{3}$. Now

$$\int_0^4 \frac{x-1}{x^2-4x-5}\,dx = \int_0^4 \left(\frac{2/3}{x-5} + \frac{1/3}{x+1}\right) dx = \left[\tfrac{2}{3}\ln|x-5| + \tfrac{1}{3}\ln|x+1|\right]_0^4$$
$$= \tfrac{2}{3}\ln 1 + \tfrac{1}{3}\ln 5 - \tfrac{2}{3}\ln 5 - \tfrac{1}{3}\ln 1 = -\tfrac{1}{3}\ln 5$$

9. $$\int \frac{x-1}{x^2-4x+5}\,dx = \int \frac{(x-2)+1}{(x-2)^2+1}\,dx = \int \left(\frac{u}{u^2+1} + \frac{1}{u^2+1}\right) du \quad [u = x - 2,\ du = dx]$$
$$= \tfrac{1}{2}\ln\left(u^2+1\right) + \tan^{-1} u + C = \tfrac{1}{2}\ln\left(x^2 - 4x + 5\right) + \tan^{-1}(x-2) + C$$

10. $$\int \frac{x}{x^4+x^2+1}\,dx = \int \frac{\frac{1}{2}\,du}{u^2+u+1} \quad [u = x^2,\ du = 2x\,dx] \quad = \frac{1}{2}\int \frac{du}{\left(u+\frac{1}{2}\right)^2 + \frac{3}{4}}$$
$$= \frac{1}{2}\int \frac{\frac{\sqrt{3}}{2}\,dv}{\frac{3}{4}\left(v^2+1\right)} \quad \left[u + \tfrac{1}{2} = \tfrac{\sqrt{3}}{2}v,\ du = \tfrac{\sqrt{3}}{2}\,dv\right] \quad = \frac{\sqrt{3}}{4}\cdot\frac{4}{3}\int \frac{dv}{v^2+1}$$
$$= \tfrac{1}{\sqrt{3}}\tan^{-1} v + C = \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{2}{\sqrt{3}}\left(x^2 + \tfrac{1}{2}\right)\right) + C$$

11. $\int \sin^3\theta\,\cos^5\theta\,d\theta = \int \cos^5\theta\,\sin^2\theta\,\sin\theta\,d\theta = -\int \cos^5\theta\,(1-\cos^2\theta)(-\sin\theta)\,d\theta$

$$= -\int u^5(1-u^2)\,du \quad \begin{bmatrix} u = \cos\theta, \\ du = -\sin\theta\,d\theta \end{bmatrix}$$
$$= \int (u^7 - u^5)\,du = \tfrac{1}{8}u^8 - \tfrac{1}{6}u^6 + C = \tfrac{1}{8}\cos^8\theta - \tfrac{1}{6}\cos^6\theta + C$$

Another solution:

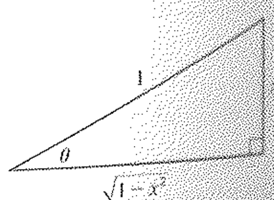
$\int \sin^3\theta\,\cos^5\theta\,d\theta = \int \sin^3\theta\,(\cos^2\theta)^2\cos\theta\,d\theta = \int \sin^3\theta\,(1-\sin^2\theta)^2\cos\theta\,d\theta$

$$= \int u^3(1-u^2)^2\,du \quad \begin{bmatrix} u = \sin\theta, \\ du = \cos\theta\,d\theta \end{bmatrix} \quad = \int u^3(1 - 2u^2 + u^4)\,du$$
$$= \int (u^3 - 2u^5 + u^7)\,du = \tfrac{1}{4}u^4 - \tfrac{1}{3}u^6 + \tfrac{1}{8}u^8 + C = \tfrac{1}{4}\sin^4\theta - \tfrac{1}{3}\sin^6\theta + \tfrac{1}{8}\sin^8\theta + C$$

12. Let $u = \cos x$. Then $du = -\sin x\,dx \quad \Rightarrow$
$\int \sin x\cos(\cos x)\,dx = -\int \cos u\,du = -\sin u + C = -\sin(\cos x) + C$.

13. Let $x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = \cos\theta\,d\theta$ and $(1-x^2)^{1/2} = \cos\theta$, so

$$\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos\theta\,d\theta}{(\cos\theta)^3} = \int \sec^2\theta\,d\theta$$
$$= \tan\theta + C = \frac{x}{\sqrt{1-x^2}} + C$$



14. Let $u = \ln x$. Then $du = dx/x \quad \Rightarrow$

$$\int \frac{\sqrt{1+\ln x}}{x\ln x}\,dx = \int \frac{\sqrt{1+u}}{u}\,du = \int \frac{v}{v^2-1}\,2v\,dv \quad \left[\text{put } v = \sqrt{1+u},\ u = v^2 - 1,\ du = 2v\,dv\right]$$
$$= 2\int \left(1 + \frac{1}{v^2-1}\right) dv = 2v + \ln\left|\frac{v-1}{v+1}\right| + C = 2\sqrt{1+\ln x} + \ln\left(\frac{\sqrt{1+\ln x} - 1}{\sqrt{1+\ln x} + 1}\right) + C$$

15. Let $u = 1 - x^2 \quad \Rightarrow \quad du = -2x\,dx$. Then

$$\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_1^{3/4} \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int_{3/4}^1 u^{-1/2}\,du = \tfrac{1}{2}\left[2u^{1/2}\right]_{3/4}^1 = \left[\sqrt{u}\right]_{3/4}^1 = 1 - \tfrac{\sqrt{3}}{2}$$

16. $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/4} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta \qquad [x = \sin\theta,\ dx = \cos\theta\,d\theta]$

$= \int_0^{\pi/4} \frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\right) - (0-0)\right] = \frac{\pi}{8} - \frac{1}{4}$

17. $\int x\sin^2 x\,dx \quad \left[\begin{matrix} u = x, & dv = \sin^2 x\,dx, \\ du = dx & v = \int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}x - \frac{1}{2}\sin x\cos x \end{matrix}\right]$

$= \frac{1}{2}x^2 - \frac{1}{2}x\sin x\cos x - \int\left(\frac{1}{2}x - \frac{1}{2}\sin x\cos x\right)dx$

$= \frac{1}{2}x^2 - \frac{1}{2}x\sin x\cos x - \frac{1}{4}x^2 + \frac{1}{4}\sin^2 x + C = \frac{1}{4}x^2 - \frac{1}{2}x\sin x\cos x + \frac{1}{4}\sin^2 x + C$

Note: $\int \sin x\cos x\,dx = \int s\,ds = \frac{1}{2}s^2 + C$ [where $s = \sin x$, $ds = \cos x\,dx$].

A slightly different method is to write $\int x\sin^2 x\,dx = \int x\cdot\frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx$. If we evaluate the second integral by parts, we arrive at the equivalent answer $\frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$.

18. Let $u = e^{2t}$, $du = 2e^{2t}\,dt$. Then

$$\int \frac{e^{2t}}{1+e^{4t}}\,dt = \int \frac{\frac{1}{2}(2e^{2t})\,dt}{1+(e^{2t})^2} = \int \frac{\frac{1}{2}du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}(e^{2t}) + C.$$

19. Let $u = e^x$. Then $\int e^{x+e^x}\,dx = \int e^{e^x}e^x\,dx = \int e^u\,du = e^u + C = e^{e^x} + C$.

20. Let $u = \sqrt[3]{x}$. Then $x = u^3 \Rightarrow \int e^{\sqrt[3]{x}}\,dx = \int e^u\cdot 3u^2\,du$. Now use parts: let $w = u^2$, $dv = e^u\,du \Rightarrow dw = 2u\,du$, $v = e^u \Rightarrow 3\int e^u u^2\,du = 3\left(u^2e^u - 2\int ue^u\,du\right)$. Now use parts again with $W = u$, $dV = e^u\,du$ to get $\int e^u\,3u^2\,du = e^u(3u^2 - 6u + 6) + C = 3e^{\sqrt[3]{x}}\left(x^{2/3} - 2\sqrt[3]{x} + 2\right) + C$.

21. Integrate by parts three times, first with $u = t^3$, $dv = e^{-2t}\,dt$:

$$\int t^3e^{-2t}\,dt = -\tfrac{1}{2}t^3e^{-2t} + \tfrac{1}{2}\int 3t^2e^{-2t}\,dt = -\tfrac{1}{2}t^3e^{-2t} - \tfrac{3}{4}t^2e^{-2t} + \tfrac{1}{2}\int 3te^{-2t}\,dt$$

$$= -e^{-2t}\left[\tfrac{1}{2}t^3 + \tfrac{3}{4}t^2\right] - \tfrac{3}{4}te^{-2t} + \tfrac{3}{4}\int e^{-2t}\,dt = -e^{-2t}\left[\tfrac{1}{2}t^3 + \tfrac{3}{4}t^2 + \tfrac{3}{4}t + \tfrac{3}{8}\right] + C$$

$$= -\tfrac{1}{8}e^{-2t}\left(4t^3 + 6t^2 + 6t + 3\right) + C$$

22. Integrate by parts: $u = \sin^{-1}x$, $dv = x\,dx \Rightarrow du = \left(1/\sqrt{1-x^2}\right)dx$, $v = \frac{1}{2}x^2$, so

$$\int x\sin^{-1}x\,dx = \tfrac{1}{2}x^2\sin^{-1}x - \frac{1}{2}\int\frac{x^2\,dx}{\sqrt{1-x^2}} = \tfrac{1}{2}x^2\sin^{-1}x - \frac{1}{2}\int\frac{\sin^2\theta\cos\theta\,d\theta}{\cos\theta} \quad \left[\begin{matrix}\text{where } x = \sin\theta \\ \text{for } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\end{matrix}\right]$$

$$= \tfrac{1}{2}x^2\sin^{-1}x - \tfrac{1}{4}\int(1-\cos 2\theta)\,d\theta = \tfrac{1}{2}x^2\sin^{-1}x - \tfrac{1}{4}(\theta - \sin\theta\cos\theta) + C$$

$$= \tfrac{1}{2}x^2\sin^{-1}x - \tfrac{1}{4}\left[\sin^{-1}x - x\sqrt{1-x^2}\right] + C = \tfrac{1}{4}\left[(2x^2-1)\sin^{-1}x + x\sqrt{1-x^2}\right] + C$$

23. Let $u = 1 + \sqrt{x}$. Then $x = (u-1)^2$, $dx = 2(u-1)\,du \Rightarrow$

$$\int_0^1\left(1+\sqrt{x}\right)^8 dx = \int_1^2 u^8\cdot 2(u-1)\,du = 2\int_1^2\left(u^9 - u^8\right)du = \left[\tfrac{1}{5}u^{10} - 2\cdot\tfrac{1}{9}u^9\right]_1^2$$

$$= \tfrac{1024}{5} - \tfrac{1024}{9} - \tfrac{1}{5} + \tfrac{2}{9} = \tfrac{4097}{45}$$

24. Let $u = \ln(x^2 - 1)$, $dv = dx \Leftrightarrow du = \dfrac{2x}{x^2 - 1}$, $v = x$. Then

$$\begin{aligned}\int \ln(x^2-1)\,dx &= x\ln(x^2-1) - \int \frac{2x^2}{x^2-1}\,dx = x\ln(x^2-1) - \int \left[2 + \frac{2}{(x-1)(x+1)}\right] dx \\ &= x\ln(x^2-1) - \int \left[2 + \frac{1}{x-1} - \frac{1}{x+1}\right] dx \\ &= x\ln(x^2-1) - 2x - \ln|x-1| + \ln|x+1| + C\end{aligned}$$

25. $\dfrac{3x^2-2}{x^2-2x-8} = 3 + \dfrac{6x+22}{(x-4)(x+2)} = 3 + \dfrac{A}{x-4} + \dfrac{B}{x+2} \Rightarrow 6x + 22 = A(x+2) + B(x-4)$. Setting $x = 4$ gives $46 = 6A$, so $A = \frac{23}{3}$. Setting $x = -2$ gives $10 = -6B$, so $B = -\frac{5}{3}$. Now
$\displaystyle\int \frac{3x^2-2}{x^2-2x-8}\,dx = \int \left(3 + \frac{23/3}{x-4} - \frac{5/3}{x+2}\right) dx = 3x + \tfrac{23}{3}\ln|x-4| - \tfrac{5}{3}\ln|x+2| + C.$

26. $\displaystyle\int \frac{3x^2-2}{x^3-2x-8}\,dx = \int \frac{du}{u} \quad \left[\begin{array}{l} u = x^3 - 2x - 8, \\ du = (3x^2-2)\,dx \end{array}\right] = \ln|u| + C = \ln|x^3 - 2x - 8| + C$

27. Let $u = \ln(\sin x)$. Then $du = \cot x\,dx \Rightarrow \int \cot x \ln(\sin x)\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}[\ln(\sin x)]^2 + C$.

28. $\int \sin\sqrt{at}\,dt = \int \sin u \cdot \frac{2}{a}u\,du \quad [u = \sqrt{at},\ u^2 = at,\ 2u\,du = a\,dt] \quad = \frac{2}{a}\int u \sin u\,du$
$= \frac{2}{a}[-u\cos u + \sin u] + C$ [integration by parts] $= -\frac{2}{a}\sqrt{at}\,\cos\sqrt{at} + \frac{2}{a}\sin\sqrt{at} + C$
$= -2\sqrt{\frac{t}{a}}\,\cos\sqrt{at} + \frac{2}{a}\sin\sqrt{at} + C$

29. $\displaystyle\int_0^5 \frac{3w-1}{w+2}\,dw = \int_0^5 \left(3 - \frac{7}{w+2}\right) dw = \Big[3w - 7\ln|w+2|\Big]_0^5$
$= 15 - 7\ln 7 + 7\ln 2 = 15 + 7(\ln 2 - \ln 7) = 15 + 7\ln\frac{2}{7}$

30. $x^2 - 4x < 0$ on $[0, 4]$, so
$\int_{-2}^{2} |x^2 - 4x|\,dx = \int_{-2}^{0}(x^2-4x)\,dx + \int_0^2 (4x - x^2)\,dx = \left[\frac{1}{3}x^3 - 2x^2\right]_{-2}^{0} + \left[2x^2 - \frac{1}{3}x^3\right]_0^2$
$= 0 - \left(-\frac{8}{3} - 8\right) + \left(8 - \frac{8}{3}\right) - 0 = 16$

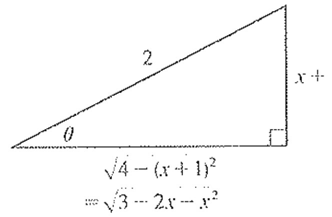
31. As in Example 5,

$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}}$$
$$= \sin^{-1}x - \sqrt{1-x^2} + C$$

Another method: Substitute $u = \sqrt{(1+x)/(1-x)}$.

32. $\displaystyle\int \frac{\sqrt{2x-1}}{2x+3}\,dx = \int \frac{u \cdot u\,du}{u^2+4} \quad \left[\begin{array}{c} u = \sqrt{2x-1},\ 2x+3 = u^2+4, \\ u^2 = 2x-1,\ u\,du = dx \end{array}\right] = \int \left(1 - \frac{4}{u^2+4}\right) du$
$= u - 4 \cdot \frac{1}{2}\tan^{-1}\left(\frac{1}{2}u\right) + C = \sqrt{2x-1} - 2\tan^{-1}\left(\frac{1}{2}\sqrt{2x-1}\right) + C$

33. $3 - 2x - x^2 = -(x^2 + 2x + 1) + 4 = 4 - (x + 1)^2$. Let $x + 1 = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\, d\theta$ and



$$\begin{aligned}\int \sqrt{3 - 2x - x^2}\, dx &= \int \sqrt{4 - (x+1)^2}\, dx = \int \sqrt{4 - 4\sin^2\theta}\; 2\cos\theta\, d\theta \\ &= 4\int \cos^2\theta\, d\theta = 2\int (1 + \cos 2\theta)\, d\theta \\ &= 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C \\ &= 2\sin^{-1}\left(\frac{x+1}{2}\right) + 2\cdot\frac{x+1}{2}\cdot\frac{\sqrt{3 - 2x - x^2}}{2} + C \\ &= 2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{x+1}{2}\sqrt{3 - 2x - x^2} + C\end{aligned}$$

34. $$\begin{aligned}\int_{\pi/4}^{\pi/2} \frac{1 + 4\cot x}{4 - \cot x}\, dx &= \int_{\pi/4}^{\pi/2} \left[\frac{(1 + 4\cos x/\sin x)}{(4 - \cos x/\sin x)}\cdot\frac{\sin x}{\sin x}\right] dx = \int_{\pi/4}^{\pi/2} \frac{\sin x + 4\cos x}{4\sin x - \cos x}\, dx \\ &= \int_{3/\sqrt{2}}^{4} \frac{1}{u}\, du \quad \begin{bmatrix} u = 4\sin x - \cos x, \\ du = (4\cos x + \sin x)\, dx \end{bmatrix} \quad = \Big[\ln|u|\Big]_{3/\sqrt{2}}^{4} \\ &= \ln 4 - \ln\frac{3}{\sqrt{2}} = \ln\frac{4}{3/\sqrt{2}} = \ln\left(\frac{4}{3}\sqrt{2}\right)\end{aligned}$$

35. Because $f(x) = x^8 \sin x$ is the product of an even function and an odd function, it is odd. Therefore, $\int_{-1}^{1} x^8 \sin x\, dx = 0$ [by (5.5.7)(b)].

36. $\sin 4x \cos 3x = \frac{1}{2}(\sin x + \sin 7x)$ by Formula 7.2.2(a), so

$$\int \sin 4x \cos 3x\, dx = \tfrac{1}{2}\int (\sin x + \sin 7x)\, dx = \tfrac{1}{2}\left[-\cos x - \tfrac{1}{7}\cos 7x\right] + C = -\tfrac{1}{2}\cos x - \tfrac{1}{14}\cos 7x + C.$$

37. $$\begin{aligned}\int_0^{\pi/4} \cos^2\theta \tan^2\theta\, d\theta &= \int_0^{\pi/4} \sin^2\theta\, d\theta = \int_0^{\pi/4} \tfrac{1}{2}(1 - \cos 2\theta)\, d\theta = \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/4} \\ &= \left(\tfrac{\pi}{8} - \tfrac{1}{4}\right) - (0 - 0) = \tfrac{\pi}{8} - \tfrac{1}{4}\end{aligned}$$

38. $$\begin{aligned}\int_0^{\pi/4} \tan^5\theta \sec^3\theta\, d\theta &= \int_0^{\pi/4} (\tan^2\theta)^2 \sec^2\theta \cdot \sec\theta\tan\theta\, d\theta = \int_1^{\sqrt{2}} (u^2 - 1)^2 u^2\, du \quad \begin{bmatrix} u = \sec\theta \\ du = \sec\theta\tan\theta\, d\theta \end{bmatrix} \\ &= \int_1^{\sqrt{2}} (u^6 - 2u^4 + u^2)\, du = \left[\tfrac{1}{7}u^7 - \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3\right]_1^{\sqrt{2}} \\ &= \left(\tfrac{8}{7}\sqrt{2} - \tfrac{8}{5}\sqrt{2} + \tfrac{2}{3}\sqrt{2}\right) - \left(\tfrac{1}{7} - \tfrac{2}{5} + \tfrac{1}{3}\right) = \tfrac{22}{105}\sqrt{2} - \tfrac{8}{105} = \tfrac{2}{105}(11\sqrt{2} - 4)\end{aligned}$$

39. Let $u = 1 - x^2$. Then $du = -2x\, dx \quad \Rightarrow$

$$\begin{aligned}\int \frac{x\, dx}{1 - x^2 + \sqrt{1 - x^2}} &= -\frac{1}{2}\int \frac{du}{u + \sqrt{u}} = -\int \frac{v\, dv}{v^2 + v} \quad [v = \sqrt{u},\ u = v^2,\ du = 2v\, dv] \\ &= -\int \frac{dv}{v + 1} = -\ln|v + 1| + C = -\ln\left(\sqrt{1 - x^2} + 1\right) + C\end{aligned}$$

40. $4y^2 - 4y - 3 = (2y - 1)^2 - 2^2$, so let $u = 2y - 1 \quad \Rightarrow \quad du = 2\, dy$. Thus,

$$\begin{aligned}\int \frac{dy}{\sqrt{4y^2 - 4y - 3}} &= \int \frac{dy}{\sqrt{(2y - 1)^2 - 2^2}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2 - 2^2}} \\ &= \tfrac{1}{2}\ln\left|u + \sqrt{u^2 - 2^2}\right| \quad \text{[by Formula 20 in the table in this section]} \\ &= \tfrac{1}{2}\ln\left|2y - 1 + \sqrt{4y^2 - 4y - 3}\right| + C\end{aligned}$$

41. Let $u = \theta$, $dv = \tan^2\theta\,d\theta = (\sec^2\theta - 1)\,d\theta \Rightarrow du = d\theta$ and $v = \tan\theta - \theta$. So

$$\int \theta\tan^2\theta\,d\theta = \theta(\tan\theta - \theta) - \int(\tan\theta - \theta)\,d\theta = \theta\tan\theta - \theta^2 - \ln|\sec\theta| + \tfrac{1}{2}\theta^2 + C$$
$$= \theta\tan\theta - \tfrac{1}{2}\theta^2 - \ln|\sec\theta| + C$$

42. Integrate by parts with $u = \tan^{-1}x$, $dv = x^2\,dx \Rightarrow du = dx/(1+x^2)$, $v = \frac{1}{3}x^3$:

$$\int x^2\tan^{-1}x\,dx = \tfrac{1}{3}x^3\tan^{-1}x - \int\frac{x^3}{3}\frac{dx}{1+x^2} = \tfrac{1}{3}x^3\tan^{-1}x - \frac{1}{3}\int\left[x - \frac{x}{x^2+1}\right]dx$$
$$= \tfrac{1}{3}x^3\tan^{-1}x - \tfrac{1}{6}x^2 + \tfrac{1}{6}\ln(x^2+1) + C$$

43. Let $u = 1 + e^x$, so that $du = e^x\,dx$. Then
$\int e^x\sqrt{1+e^x}\,dx = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$.
Or: Let $u = \sqrt{1+e^x}$, so that $u^2 = 1 + e^x$ and $2u\,du = e^x\,dx$. Then
$\int e^x\sqrt{1+e^x}\,dx = \int u\cdot 2u\,du = \int 2u^2du = \frac{2}{3}u^3 + C = \frac{2}{3}(1+e^x)^{3/2} + C$.

44. Let $u = \sqrt{1+e^x}$. Then $u^2 = 1 + e^x$, $2u\,du = e^x\,dx = (u^2-1)\,dx$, and $dx = \dfrac{2u}{u^2-1}\,du$, so

$$\int\sqrt{1+e^x}\,dx = \int u\cdot\frac{2u}{u^2-1}\,du = \int\frac{2u^2}{u^2-1}\,du = \int\left(2 + \frac{2}{u^2-1}\right)du = \int\left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right)du$$
$$= 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{1+e^x} + \ln\left(\sqrt{1+e^x} - 1\right) - \ln\left(\sqrt{1+e^x} + 1\right) + C$$

45. Let $t = x^3$. Then $dt = 3x^2\,dx \Rightarrow I = \int x^5e^{-x^3}\,dx = \frac{1}{3}\int te^{-t}\,dt$. Now integrate by parts with $u = t$, $dv = e^{-t}\,dt$: $I = -\frac{1}{3}te^{-t} + \frac{1}{3}\int e^{-t}\,dt = -\frac{1}{3}te^{-t} - \frac{1}{3}e^{-t} + C = -\frac{1}{3}e^{-x^3}(x^3+1) + C$.

46. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \Rightarrow$

$$\int\frac{1+e^x}{1-e^x}\,dx = \int\frac{(1+u)\,du}{(1-u)u} = -\int\frac{(u+1)\,du}{(u-1)u} = -\int\left(\frac{2}{u-1} - \frac{1}{u}\right)du$$
$$= \ln|u| - 2\ln|u-1| + C = \ln e^x - 2\ln|e^x - 1| + C = x - 2\ln|e^x - 1| + C$$

47. $$\int\frac{x+a}{x^2+a^2}\,dx = \frac{1}{2}\int\frac{2x\,dx}{x^2+a^2} + a\int\frac{dx}{x^2+a^2} = \tfrac{1}{2}\ln(x^2+a^2) + a\cdot\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$
$$= \ln\sqrt{x^2+a^2} + \tan^{-1}(x/a) + C$$

48. Let $u = x^2$. Then $du = 2x\,dx \Rightarrow$
$$\int\frac{x\,dx}{x^4-a^4} = \int\frac{\frac{1}{2}\,du}{u^2-(a^2)^2} = \frac{1}{4a^2}\ln\left|\frac{u-a^2}{u+a^2}\right| + C = \frac{1}{4a^2}\ln\left|\frac{x^2-a^2}{x^2+a^2}\right| + C.$$

49. Let $u = \sqrt{4x+1} \Rightarrow u^2 = 4x + 1 \Rightarrow 2u\,du = 4\,dx \Rightarrow dx = \frac{1}{2}u\,du$. So

$$\int\frac{1}{x\sqrt{4x+1}}\,dx = \int\frac{\frac{1}{2}u\,du}{\frac{1}{4}(u^2-1)\,u} = 2\int\frac{du}{u^2-1} = 2\left(\tfrac{1}{2}\right)\ln\left|\frac{u-1}{u+1}\right| + C \quad \text{[by Formula 19]}$$
$$= \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C$$

50. As in Exercise 49, let $u = \sqrt{4x+1}$. Then $\displaystyle\int \frac{dx}{x^2\sqrt{4x+1}} = \int \frac{\frac{1}{2}u\,du}{\left[\frac{1}{4}(u^2-1)\right]^2 u} = 8\int \frac{du}{(u^2-1)^2}$. Now

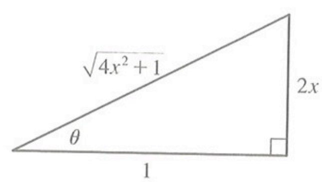
$$\frac{1}{(u^2-1)^2} = \frac{1}{(u+1)^2(u-1)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2} \quad\Rightarrow$$

$1 = A(u+1)(u-1)^2 + B(u-1)^2 + C(u-1)(u+1)^2 + D(u+1)^2$. $u = 1 \;\Rightarrow\; D = \frac{1}{4}$, $u = -1 \;\Rightarrow$ $B = \frac{1}{4}$. Equating coefficients of u^3 gives $A + C = 0$, and equating coefficients of 1 gives $1 = A + B - C + D$ $\Rightarrow\; 1 = A + \frac{1}{4} - C + \frac{1}{4} \;\Rightarrow\; \frac{1}{2} = A - C$. So $A = \frac{1}{4}$ and $C = -\frac{1}{4}$. Therefore,

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{4x+1}} &= 8\int \left[\frac{1/4}{u+1} + \frac{1/4}{(u+1)^2} + \frac{-1/4}{u-1} + \frac{1/4}{(u-1)^2}\right] du \\
&= \int \left[\frac{2}{u+1} + 2(u+1)^{-2} - \frac{2}{u-1} + 2(u-1)^{-2}\right] du \\
&= 2\ln|u+1| - \frac{2}{u+1} - 2\ln|u-1| - \frac{2}{u-1} + C \\
&= 2\ln\left(\sqrt{4x+1}+1\right) - \frac{2}{\sqrt{4x+1}+1} - 2\ln\left|\sqrt{4x+1}-1\right| - \frac{2}{\sqrt{4x+1}-1} + C
\end{aligned}$$

51. Let $2x = \tan\theta \;\Rightarrow\; x = \frac{1}{2}\tan\theta$, $dx = \frac{1}{2}\sec^2\theta\,d\theta$, $\sqrt{4x^2+1} = \sec\theta$, so

$$\begin{aligned}
\int \frac{dx}{x\sqrt{4x^2+1}} &= \int \frac{\frac{1}{2}\sec^2\theta\,d\theta}{\frac{1}{2}\tan\theta\sec\theta} = \int \frac{\sec\theta}{\tan\theta}\,d\theta = \int \csc\theta\,d\theta \\
&= -\ln|\csc\theta + \cot\theta| + C \qquad [\text{or } \ln|\csc\theta - \cot\theta| + C] \\
&= -\ln\left|\frac{\sqrt{4x^2+1}}{2x} + \frac{1}{2x}\right| + C \qquad \left[\text{or } \ln\left|\frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x}\right| + C\right]
\end{aligned}$$



52. Let $u = x^2$. Then $du = 2x\,dx \;\Rightarrow$

$$\begin{aligned}
\int \frac{dx}{x(x^4+1)} &= \int \frac{x\,dx}{x^2(x^4+1)} = \frac{1}{2}\int \frac{du}{u(u^2+1)} = \frac{1}{2}\int \left[\frac{1}{u} - \frac{u}{u^2+1}\right] du = \tfrac{1}{2}\ln|u| - \tfrac{1}{4}\ln(u^2+1) + C \\
&= \tfrac{1}{2}\ln(x^2) - \tfrac{1}{4}\ln(x^4+1) + C = \tfrac{1}{4}\left[\ln(x^4) - \ln(x^4+1)\right] + C = \frac{1}{4}\ln\left(\frac{x^4}{x^4+1}\right) + C
\end{aligned}$$

Or: Write $I = \displaystyle\int \frac{x^3\,dx}{x^4(x^4+1)}$ and let $u = x^4$.

53. $\displaystyle\int x^2\sinh(mx)dx = \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\int x\cosh(mx)\,dx$ $\qquad\left[\begin{matrix} u = x^2, & dv = \sinh(mx)\,dx, \\ du = 2x\,dx & v = \frac{1}{m}\cosh(mx) \end{matrix}\right]$

$$\begin{aligned}
&= \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\left(\tfrac{1}{m}x\sinh(mx) - \frac{1}{m}\textstyle\int \sinh(mx)\,dx\right) \qquad \left[\begin{matrix} U = x, & dV = \cosh(mx)\,dx, \\ dU = dx & V = \frac{1}{m}\sinh(mx) \end{matrix}\right] \\
&= \frac{1}{m}x^2\cosh(mx) - \frac{2}{m^2}x\sinh(mx) + \frac{2}{m^3}\cosh(mx) + C
\end{aligned}$$

54. $\int (x + \sin x)^2\,dx = \int \left(x^2 + 2x\sin x + \sin^2 x\right) dx = \frac{1}{3}x^3 + 2(\sin x - x\cos x) + \frac{1}{2}(x - \sin x\cos x) + C$

$$= \tfrac{1}{3}x^3 + \tfrac{1}{2}x + 2\sin x - \tfrac{1}{2}\sin x\cos x - 2x\cos x + C$$

55. Let $u = \sqrt{x+1}$. Then $x = u^2 - 1 \quad \Rightarrow$

$$\int \frac{dx}{x+4+4\sqrt{x+1}} = \int \frac{2u\,du}{u^2+3+4u} = \int \left[\frac{-1}{u+1} + \frac{3}{u+3}\right] du$$
$$= 3\ln|u+3| - \ln|u+1| + C = 3\ln\left(\sqrt{x+1}+3\right) - \ln\left(\sqrt{x+1}+1\right) + C$$

56. Let $t = \sqrt{x^2-1}$. Then $dt = \left(x/\sqrt{x^2-1}\right)dx$, $x^2 - 1 = t^2$, $x = \sqrt{t^2+1}$, so

$I = \displaystyle\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx = \int \ln\sqrt{t^2+1}\,dt = \frac{1}{2}\int \ln\left(t^2+1\right)dt$. Now use parts with $u = \ln(t^2+1)$, $dv = dt$:

$$I = \tfrac{1}{2}t\ln(t^2+1) - \int \frac{t^2}{t^2+1}\,dt = \tfrac{1}{2}t\ln(t^2+1) - \int \left[1 - \frac{1}{t^2+1}\right] dt$$
$$= \tfrac{1}{2}t\ln(t^2+1) - t + \tan^{-1}t + C = \sqrt{x^2-1}\,\ln x - \sqrt{x^2-1} + \tan^{-1}\sqrt{x^2-1} + C$$

Another method: First integrate by parts with $u = \ln x$, $dv = \left(x/\sqrt{x^2-1}\right)dx$ and then use substitution $\left(x = \sec\theta \text{ or } u = \sqrt{x^2-1}\right)$.

57. Let $u = \sqrt[3]{x+c}$. Then $x = u^3 - c \quad \Rightarrow$

$$\textstyle\int x\,\sqrt[3]{x+c}\,dx = \int (u^3 - c)u \cdot 3u^2\,du = 3\int \left(u^6 - cu^3\right)du = \frac{3}{7}u^7 - \frac{3}{4}cu^4 + C$$
$$= \tfrac{3}{7}(x+c)^{7/3} - \tfrac{3}{4}c(x+c)^{4/3} + C$$

58. Integrate by parts with $u = \ln(1+x)$, $dv = x^2\,dx \quad \Rightarrow \quad du = dx/(1+x)$, $v = \frac{1}{3}x^3$:

$$\int x^2\ln(1+x)\,dx = \tfrac{1}{3}x^3\ln(1+x) - \int \frac{x^3\,dx}{3(1+x)} = \tfrac{1}{3}x^3\ln(1+x) - \frac{1}{3}\int\left(x^2 - x + 1 - \frac{1}{x+1}\right)dx$$
$$= \tfrac{1}{3}x^3\ln(1+x) - \tfrac{1}{9}x^3 + \tfrac{1}{6}x^2 - \tfrac{1}{3}x + \tfrac{1}{3}\ln(1+x) + C$$

59. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \quad \Rightarrow$

$$\int \frac{dx}{e^{3x} - e^x} = \int \frac{du/u}{u^3 - u} = \int \frac{du}{(u-1)u^2(u+1)} = \int \left[\frac{1/2}{u-1} - \frac{1}{u^2} - \frac{1/2}{u+1}\right] du$$
$$= \frac{1}{u} + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C = e^{-x} + \frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right| + C$$

60. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2\,du \quad \Rightarrow$

$$\int \frac{dx}{x+\sqrt[3]{x}} = \int \frac{3u^2\,du}{u^3+u} = \frac{3}{2}\int \frac{2u\,du}{u^2+1} = \tfrac{3}{2}\ln\left(u^2+1\right) + C = \tfrac{3}{2}\ln\left(x^{2/3}+1\right) + C.$$

61. Let $u = x^5$. Then $du = 5x^4\,dx \quad \Rightarrow$

$$\int \frac{x^4\,dx}{x^{10}+16} = \int \frac{\frac{1}{5}\,du}{u^2+16} = \tfrac{1}{5}\cdot\tfrac{1}{4}\tan^{-1}\left(\tfrac{1}{4}u\right) + C = \tfrac{1}{20}\tan^{-1}\left(\tfrac{1}{4}x^5\right) + C.$$

62. Let $u = x + 1$. Then $du = dx \quad \Rightarrow$

$$\int \frac{x^3}{(x+1)^{10}}\,dx = \int \frac{(u-1)^3}{u^{10}}\,du = \int \left(u^{-7} - 3u^{-8} + 3u^{-9} - u^{-10}\right) du$$
$$= -\tfrac{1}{6}u^{-6} + \tfrac{3}{7}u^{-7} - \tfrac{3}{8}u^{-8} + \tfrac{1}{9}u^{-9} + C$$
$$= (x+1)^{-9}\left[-\tfrac{1}{6}(x+1)^3 + \tfrac{3}{7}(x+1)^2 - \tfrac{3}{8}(x+1) + \tfrac{1}{9}\right] + C$$

63. Let $y = \sqrt{x}$ so that $dy = \dfrac{1}{2\sqrt{x}}\,dx \;\Rightarrow\; dx = 2\sqrt{x}\,dy = 2y\,dy$. Then

$$\int \sqrt{x}\,e^{\sqrt{x}}\,dx = \int ye^{y}(2y\,dy) = \int 2y^2e^y\,dy \qquad \begin{bmatrix} u = 2y^2, & dv = e^y\,dy, \\ du = 4y\,dy & v = e^y \end{bmatrix}$$

$$= 2y^2e^y - \int 4ye^y\,dy \qquad \begin{bmatrix} U = 4y, & dV = e^y\,dy, \\ dU = 4\,dy & V = e^y \end{bmatrix}$$

$$= 2y^2e^y - \left(4ye^y - \int 4e^y\,dy\right) = 2y^2e^y - 4ye^y + 4e^y + C$$

$$= 2(y^2 - 2y + 2)e^y + C = 2(x - 2\sqrt{x} + 2)\,e^{\sqrt{x}} + C$$

64. Let $u = \tan x$. Then

$$\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)\,dx}{\sin x \cos x} = \int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\tan x}\sec^2 x\,dx = \int_1^{\sqrt{3}} \frac{\ln u}{u}\,du$$

$$= \left[\tfrac{1}{2}(\ln u)^2\right]_1^{\sqrt{3}} = \tfrac{1}{2}\left(\ln\sqrt{3}\right)^2 = \tfrac{1}{8}(\ln 3)^2$$

65. $$\int \frac{dx}{\sqrt{x+1} + \sqrt{x}} = \int \left(\frac{1}{\sqrt{x+1} + \sqrt{x}} \cdot \frac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x+1} - \sqrt{x}}\right)dx = \int \left(\sqrt{x+1} - \sqrt{x}\right)dx$$

$$= \tfrac{2}{3}\left[(x+1)^{3/2} - x^{3/2}\right] + C$$

66. $$\int \frac{u^3+1}{u^3-u^2}\,du = \int \left[1 + \frac{u^2+1}{(u-1)u^2}\right]du = u + \int \left[\frac{2}{u-1} - \frac{1}{u} - \frac{1}{u^2}\right]du = u + 2\ln|u-1| - \ln|u| + \frac{1}{u} + C.$$

Thus,

$$\int_2^3 \frac{u^3+1}{u^3-u^2}\,du = \left[u + 2\ln(u-1) - \ln u + \frac{1}{u}\right]_2^3 = \left(3 + 2\ln 2 - \ln 3 + \tfrac{1}{3}\right) - \left(2 + 2\ln 1 - \ln 2 + \tfrac{1}{2}\right)$$

$$= 1 + 3\ln 2 - \ln 3 - \tfrac{1}{6} = \tfrac{5}{6} + \ln\tfrac{8}{3}$$

67. Let $u = \sqrt{t}$. Then $du = dt/\left(2\sqrt{t}\right) \;\Rightarrow$

$$\int_1^3 \frac{\arctan\sqrt{t}}{\sqrt{t}}\,dt = \int_1^{\sqrt{3}} \tan^{-1}u\,(2\,du) = 2\left[u\tan^{-1}u - \tfrac{1}{2}\ln(1+u^2)\right]_1^{\sqrt{3}} \qquad \text{[Example 5 in Section 7.1]}$$

$$= 2\left[\left(\sqrt{3}\tan^{-1}\sqrt{3} - \tfrac{1}{2}\ln 4\right) - \left(\tan^{-1}1 - \tfrac{1}{2}\ln 2\right)\right]$$

$$= 2\left[\left(\sqrt{3}\cdot\tfrac{\pi}{3} - \ln 2\right) - \left(\tfrac{\pi}{4} - \tfrac{1}{2}\ln 2\right)\right] = \tfrac{2}{3}\sqrt{3}\,\pi - \tfrac{1}{2}\pi - \ln 2$$

68. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \;\Rightarrow$

$$\int \frac{dx}{1 + 2e^x - e^{-x}} = \int \frac{du/u}{1 + 2u - 1/u} = \int \frac{du}{2u^2 + u - 1} = \int \left[\frac{2/3}{2u-1} - \frac{1/3}{u+1}\right]du$$

$$= \tfrac{1}{3}\ln|2u-1| - \tfrac{1}{3}\ln|u+1| + C = \tfrac{1}{3}\ln\left|(2e^x - 1)/(e^x + 1)\right| + C$$

69. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \;\Rightarrow$

$$\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{u^2}{1+u}\frac{du}{u} = \int \frac{u}{1+u}\,du = \int \left(1 - \frac{1}{1+u}\right)du$$

$$= u - \ln|1+u| + C = e^x - \ln(1+e^x) + C$$

70. Use parts with $u = \ln(x+1)$, $dv = dx/x^2$:

$$\int \frac{\ln(x+1)}{x^2}\,dx = -\frac{1}{x}\ln(x+1) + \int \frac{dx}{x(x+1)} = -\frac{1}{x}\ln(x+1) + \int \left[\frac{1}{x} - \frac{1}{x+1}\right]dx$$

$$= -\tfrac{1}{x}\ln(x+1) + \ln|x| - \ln(x+1) + C = -\left(1 + \tfrac{1}{x}\right)\ln(x+1) + \ln|x| + C$$

71. $\dfrac{x}{x^4+4x^2+3} = \dfrac{x}{(x^2+3)(x^2+1)} = \dfrac{Ax+B}{x^2+3} + \dfrac{Cx+D}{x^2+1} \Rightarrow$

$$x = (Ax+B)(x^2+1) + (Cx+D)(x^2+3) = (Ax^3+Bx^2+Ax+B) + (Cx^3+Dx^2+3Cx+3D)$$
$$= (A+C)x^3 + (B+D)x^2 + (A+3C)x + (B+3D) \Rightarrow$$

$A+C=0$, $B+D=0$, $A+3C=1$, $B+3D=0 \Rightarrow A=-\frac{1}{2}$, $C=\frac{1}{2}$, $B=0$, $D=0$. Thus,

$$\int \frac{x}{x^4+4x^2+3}\,dx = \int \left(\frac{-\frac{1}{2}x}{x^2+3} + \frac{\frac{1}{2}x}{x^2+1}\right) dx$$
$$= -\tfrac{1}{4}\ln(x^2+3) + \tfrac{1}{4}\ln(x^2+1) + C \quad \text{or} \quad \frac{1}{4}\ln\left(\frac{x^2+1}{x^2+3}\right) + C$$

72. Let $u = \sqrt[6]{t}$. Then $t = u^6$, $dt = 6u^5\,du \Rightarrow$

$$\int \frac{\sqrt{t}\,dt}{1+\sqrt[3]{t}} = \int \frac{u^3 \cdot 6u^5\,du}{1+u^2} = 6\int \frac{u^8}{u^2+1}\,du = 6\int \left(u^6 - u^4 + u^2 - 1 + \frac{1}{u^2+1}\right) du$$
$$= 6\left(\tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + \tfrac{1}{3}u^3 - u + \tan^{-1} u\right) + C$$
$$= 6\left(\tfrac{1}{7}t^{7/6} - \tfrac{1}{5}t^{5/6} + \tfrac{1}{3}t^{1/2} - t^{1/6} + \tan^{-1} t^{1/6}\right) + C$$

73. $\dfrac{1}{(x-2)(x^2+4)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+4} \Rightarrow$

$1 = A(x^2+4) + (Bx+C)(x-2) = (A+B)x^2 + (C-2B)x + (4A-2C)$. So $0 = A+B = C-2B$, $1 = 4A-2C$. Setting $x=2$ gives $A=\frac{1}{8} \Rightarrow B=-\frac{1}{8}$ and $C=-\frac{1}{4}$. So

$$\int \frac{1}{(x-2)(x^2+4)}\,dx = \int \left(\frac{\frac{1}{8}}{x-2} + \frac{-\frac{1}{8}x-\frac{1}{4}}{x^2+4}\right) dx = \frac{1}{8}\int \frac{dx}{x-2} - \frac{1}{16}\int \frac{2x\,dx}{x^2+4} - \frac{1}{4}\int \frac{dx}{x^2+4}$$
$$= \tfrac{1}{8}\ln|x-2| - \tfrac{1}{16}\ln(x^2+4) - \tfrac{1}{8}\tan^{-1}(x/2) + C$$

74. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \Rightarrow$

$$\int \frac{dx}{e^x - e^{-x}} = \int \frac{e^x\,dx}{e^{2x}-1} = \int \frac{u}{u^2-1}\frac{du}{u} = \int \frac{du}{u^2-1} = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C = \frac{1}{2}\ln\left(\frac{|e^x-1|}{e^x+1}\right) + C.$$

75. $\int \sin x \sin 2x \sin 3x\,dx = \int \sin x \cdot \frac{1}{2}[\cos(2x-3x) - \cos(2x+3x)]\,dx = \frac{1}{2}\int (\sin x\cos x - \sin x\cos 5x)\,dx$
$$= \tfrac{1}{4}\int \sin 2x\,dx - \tfrac{1}{2}\int \tfrac{1}{2}[\sin(x+5x) + \sin(x-5x)]\,dx$$
$$= -\tfrac{1}{8}\cos 2x - \tfrac{1}{4}\int(\sin 6x - \sin 4x)\,dx = -\tfrac{1}{8}\cos 2x + \tfrac{1}{24}\cos 6x - \tfrac{1}{16}\cos 4x + C$$

76. $\int (x^2 - bx)\sin 2x\,dx = -\frac{1}{2}(x^2-bx)\cos 2x + \frac{1}{2}\int (2x-b)\cos 2x\,dx$

$[u = x^2 - bx,\ dv = \sin 2x\,dx,\ du = (2x-b)\,dx,\ v = -\frac{1}{2}\cos 2x]$

$$= -\tfrac{1}{2}(x^2-bx)\cos 2x + \tfrac{1}{2}\left[\tfrac{1}{2}(2x-b)\sin 2x - \int \sin 2x\,dx\right]$$

$[U = 2x-b,\ dV = \cos 2x\,dx,\ dU = 2\,dx,\ V = \frac{1}{2}\sin 2x]$

$$= -\tfrac{1}{2}(x^2-bx)\cos 2x + \tfrac{1}{4}(2x-b)\sin 2x + \tfrac{1}{4}\cos 2x + C$$

77. Let $u = x^{3/2}$ so that $u^2 = x^3$ and $du = \frac{3}{2}x^{1/2}\,dx \quad\Rightarrow\quad \sqrt{x}\,dx = \frac{2}{3}\,du$. Then

$$\int \frac{\sqrt{x}}{1+x^3}\,dx = \int \frac{\frac{2}{3}}{1+u^2}\,du = \frac{2}{3}\tan^{-1}u + C = \frac{2}{3}\tan^{-1}(x^{3/2}) + C.$$

78. $$\int \frac{\sec x\,\cos 2x}{\sin x + \sec x}\,dx = \int \frac{\sec x\,\cos 2x}{\sin x + \sec x}\cdot\frac{2\cos x}{2\cos x}\,dx = \int \frac{2\cos 2x}{2\sin x\cos x + 2}\,dx$$

$$= \int \frac{2\cos 2x}{\sin 2x + 2}\,dx = \int \frac{1}{u}\,du \qquad \begin{bmatrix} u = \sin 2x + 2, \\ du = 2\cos 2x\,dx \end{bmatrix}$$

$$= \ln|u| + C = \ln|\sin 2x + 2| + C = \ln(\sin 2x + 2) + C$$

79. Let $u = x$, $dv = \sin^2 x\cos x\,dx \quad\Rightarrow\quad du = dx$, $v = \frac{1}{3}\sin^3 x$. Then

$$\int x\sin^2 x\,\cos x\,dx = \tfrac{1}{3}x\sin^3 x - \int \tfrac{1}{3}\sin^3 x\,dx = \tfrac{1}{3}x\sin^3 x - \tfrac{1}{3}\int (1-\cos^2 x)\sin x\,dx$$

$$= \frac{1}{3}x\sin^3 x + \frac{1}{3}\int (1-y^2)\,dy \qquad \begin{bmatrix} y = \cos x, \\ dy = -\sin x\,dx \end{bmatrix}$$

$$= \tfrac{1}{3}x\sin^3 x + \tfrac{1}{3}y - \tfrac{1}{9}y^3 + C = \tfrac{1}{3}x\sin^3 x + \tfrac{1}{3}\cos x - \tfrac{1}{9}\cos^3 x + C$$

80. $$\int \frac{\sin x\,\cos x}{\sin^4 x + \cos^4 x}\,dx = \int \frac{\sin x\,\cos x}{(\sin^2 x)^2 + (\cos^2 x)^2}\,dx = \int \frac{\sin x\,\cos x}{(\sin^2 x)^2 + (1-\sin^2 x)^2}\,dx$$

$$= \int \frac{1}{u^2 + (1-u)^2}\left(\frac{1}{2}\,du\right) \qquad \begin{bmatrix} u = \sin^2 x, \\ du = 2\sin x\cos x\,dx \end{bmatrix}$$

$$= \int \frac{1}{4u^2 - 4u + 2}\,du = \int \frac{1}{(4u^2 - 4u + 1) + 1}\,du$$

$$= \int \frac{1}{(2u-1)^2 + 1}\,du = \frac{1}{2}\int \frac{1}{y^2+1}\,dy \qquad \begin{bmatrix} y = 2u - 1, \\ dy = 2\,du \end{bmatrix}$$

$$= \tfrac{1}{2}\tan^{-1}y + C = \tfrac{1}{2}\tan^{-1}(2u-1) + C = \tfrac{1}{2}\tan^{-1}(2\sin^2 x - 1) + C$$

Another solution:

$$\int \frac{\sin x\,\cos x}{\sin^4 x + \cos^4 x}\,dx = \int \frac{(\sin x\,\cos x)/\cos^4 x}{(\sin^4 x + \cos^4 x)/\cos^4 x}\,dx = \int \frac{\tan x\sec^2 x}{\tan^4 x + 1}\,dx$$

$$= \int \frac{1}{u^2+1}\left(\frac{1}{2}\,du\right) \qquad \begin{bmatrix} u = \tan^2 x, \\ du = 2\tan x\sec^2 x\,dx \end{bmatrix}$$

$$= \tfrac{1}{2}\tan^{-1}u + C = \tfrac{1}{2}\tan^{-1}(\tan^2 x) + C$$

81. The function $y = 2xe^{x^2}$ *does* have an elementary antiderivative, so we'll use this fact to help evaluate the integral.

$$\int (2x^2+1)e^{x^2}\,dx = \int 2x^2e^{x^2}\,dx + \int e^{x^2}\,dx = \int x(2xe^{x^2})\,dx + \int e^{x^2}\,dx$$

$$= xe^{x^2} - \int e^{x^2}\,dx + \int e^{x^2}\,dx \qquad \begin{bmatrix} u = x, & dv = 2xe^{x^2}\,dx, \\ du = dx & v = e^{x^2} \end{bmatrix} \qquad = xe^{x^2} + C$$