A Deeper Look at a Derivative Sketching Activity

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"It is the harmony of the diverse parts, their symmetry, their happy balance; in a word it is all that introduces order, all that gives unity, that permits us to see clearly and to comprehend at once both the ensemble and the details."

Figure: ?

"Thought is only a flash between two long nights, but this flash is everything."

The Necessity Principle

"For students to learn what we intend to teach them, they must have a need for it, where 'need' refers to **intellectual need**, not social or economic need."

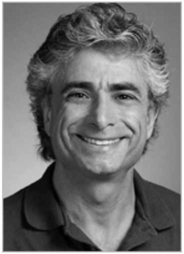


Figure: Guershon Harel

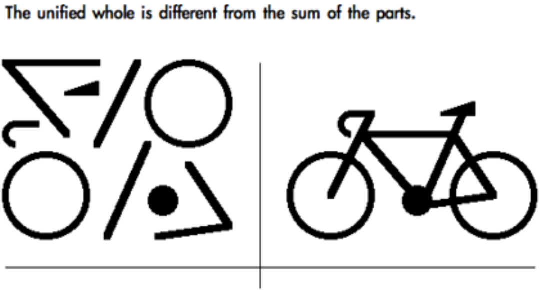
"Learning is the discovery that something is possible." - Fritz Perls (1893-1970)

Figure: Fritz Perls

Father of Gestalt Therapy - *"Lose your Mind and come to your senses."*

FLOCK-inspired Math 75 Redesign Features

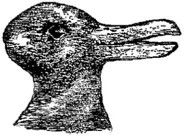
- Three lecture/problem solving days (MTW) and 1 active learning day (TH).
- Sequencing of topics based upon the 'Wholecept Resolution' perspective as much as possible.
- Active Learning 'Tactivities' mostly from http://math.colorado.edu/activecalc/
- Designated class periods having students working at boards together on 12 group quizzes.
- Students encouraged to attend Calculus Success weekly 2 hour sessions in addition to SI.



Definition

A **Wholecept** *is a gestalt-like cognitive* structure, arrangement, or pattern of mathematical phenomena so integrated as to constitute a functional unit with properties not derivable by summation of its parts.

gestalt.



Examples

in Menschengestalt →*in human form*

Examples

sich in seiner wahren Gestalt zeigen→*to show one's true colors*

Examples

Gehalt und Gestalt→*Form and Content*

Examples

Gestalt geworden→*made flesh*

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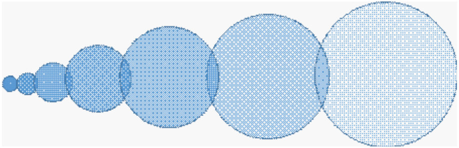
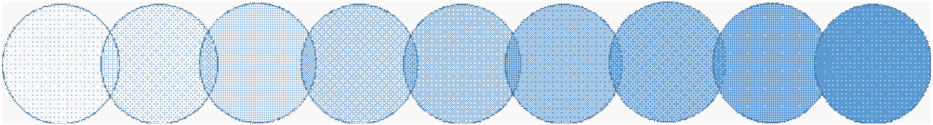
- Recognize what constitutes an intellectual need for a particular population of students, relative to the concept to be learned.
- Present students with a sequence of problems [and activities] that correspond to their intellectual needs [necessity to realize a WHOLECEPT].
- Help students elicit the [WHOLECEPT] from the problem solutions [and activities] as a shared dialogue of increased Awareness.



Unilinear Concept Formation

'Unilinear - developing or arranged serially and predictably, without deviation'



'Wholecept' Resolution

Difference Quotient Wholecept introduction on day 1

... review for the sake of review is discouraged!

They need to know 'why' they are presented with this knowledge, so it's best to tell them asap. [Necessity Principle]

Week	*M*	*T*	*W*	*TH-Tactivities*
1	1.1	1.2	1.3	Function Placemats
2	1.4	2.2	*gps*	Transformation Matching
3	2.3	2.4	*gps*	Limit Sentences
4	2.5	2.6	*gps*	**Exam 1**
5	3.1-3.2	3.3-3.4	**3.7-3.8**	Graphical Limit Laws
6	3.5/3.7	3.9/3.7	3.10/3.7	Definition of Derivative
7	*3.11*	*3.11*	*3.11*	Related Rates Solitaire
8	3.6	*gps*	*gps*	**Exam 2**
9	4.1	4.2	4.3	Derivative Matching Cards
10	4.4	4.5	*gps*	Grade this Quiz
11	4.6	4.7	*gps*	Sketching Snippets
12	4.8	4.9	*gps*	**Exam 3**
13	5.1	5.2	*gps*	Wacky Limits
14	5.3-5.4	5.5	*gps*	Definite Integral Dominoes
15	*5.5*	*5.5*	*gps*	Cups
16	*5.5*	*5.5*	*gps*	**Exam 4**

Wholecepts to Teach Early with Repetition

- Difference Quotient
- A secant average slope

- The Chain Rule - Conceptual Justifications
- Implicit Differentiation

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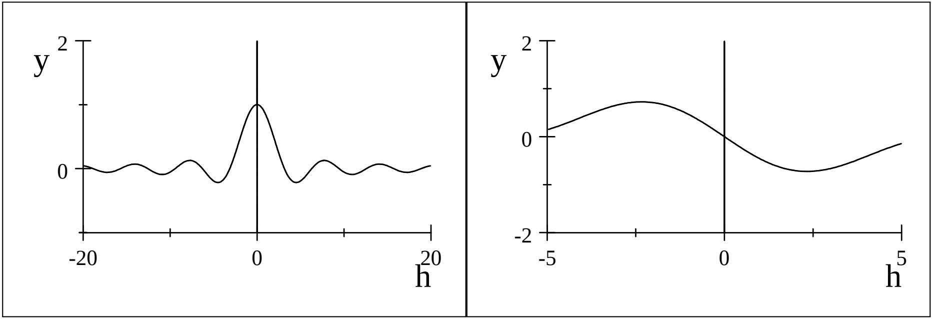


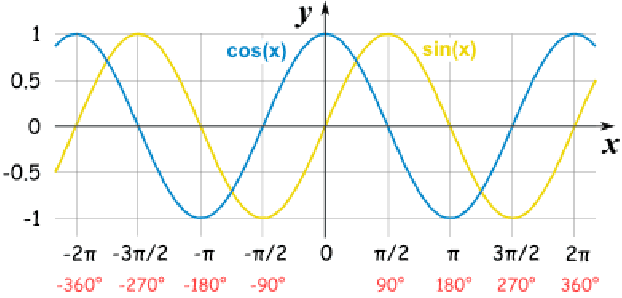
$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1 \qquad \lim_{h\to 0}\frac{\cos(h)-1}{h} = 0$$

$$(\sin x)' = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos(h)+\sin(h)\cos x-\sin x}{h} =$$

$$\lim_{h\to 0}\frac{\sin x(\cos(h)-1)+\sin(h)\cos x}{h}$$

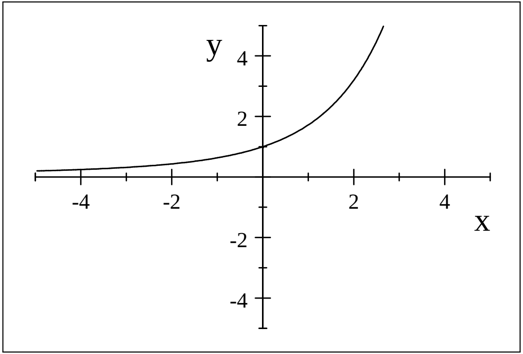
$$\lim_{h\to 0}\sin x\cdot\frac{\cos(h)-1}{h}+\cos x\cdot\frac{\sin(h)}{h} = \sin x\cdot 0+\cos x\cdot 1 = \cos x.$$

- The Chain Rule rules!
  $(\cos x)' = (\sin(x + \frac{\pi}{2}))' = \cos(x + \frac{\pi}{2}) \cdot \frac{d(x+\frac{\pi}{2})}{dx} = -\sin x.$



$(\sin x)' = (\cos(x - \frac{\pi}{2}))' = -\sin(x - \frac{\pi}{2}) = \cos x$

$(e^x)'$



$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

$$(e^x)' = \lim_{h \to 0} \frac{e^{(x+h)} - e^x}{h} = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} = e^x$$

$(lnx)\prime$ and $(x^n)'$

- Chain Rule $\rightarrow$Implicit Differentiation

$y = \ln x \rightarrow e^y = x \rightarrow$

$$(e^y)' = e^y \cdot y' = 1 \rightarrow y' = \frac{1}{e^y} = \frac{1}{x}$$

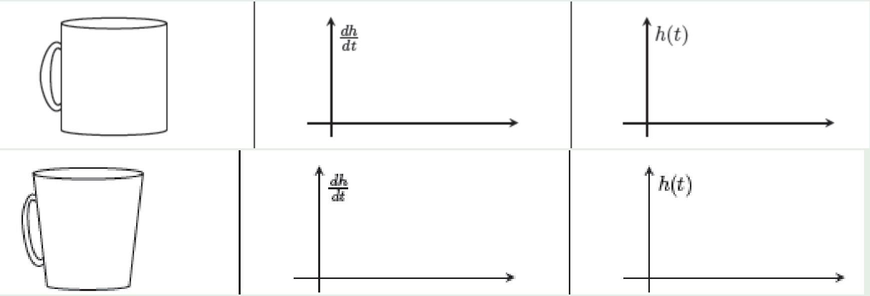
$y = x^n \rightarrow \ln y = \ln(x^n) \rightarrow$
$\ln y = n \ln x \rightarrow \frac{1}{y} \cdot y' = \frac{n}{x} \rightarrow$
$y' = \frac{n}{x} \cdot y = n\frac{x^n}{x} = nx^{n-1}$.

Activity Description

Problem

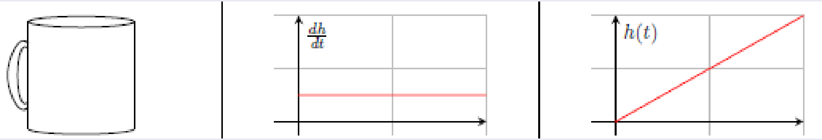
Coffee is being poured at a constant rate v into coffee cups of various shapes. Sketch rough graphs of the rate of change of the depth $h'(t)$ and of the depth $h(t)$ as a functions of time t.

Example



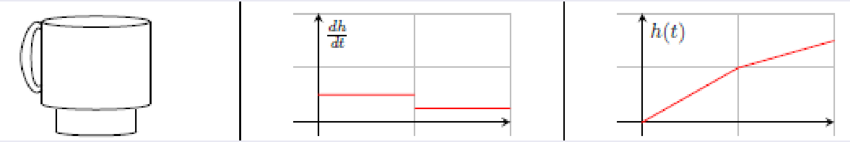
Most students produce graphs like this:

Solution



Students also tend to negotiate this type of cup:

Solution (two stacked-cylinders)

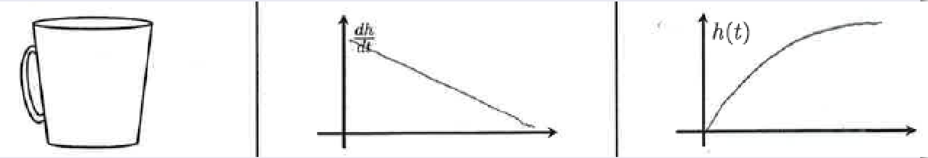


But for this cup ...

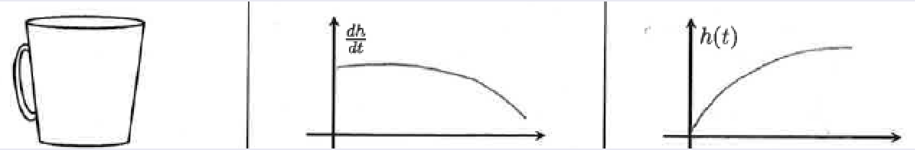
Example (inverted frustrum)

Two types of student-solutions occurred about 75% of the time last semester:

Solution (linear rate decrease)



Solution (concave down rate decrease)



Now for a deeper look ...

Example

Back to the cylindrical cup with base radius r_0, we can safely conclude that since the *volume* $V(t)$ of coffee in the cup increases at a constant rate, then so does its *depth*.

Hence, $h'(t) \equiv h$ and $h(t) = ht$ (the cup being empty initially, i.e., $h(0) = 0$).

$$V(t) = \pi r_0^2 h(t).$$

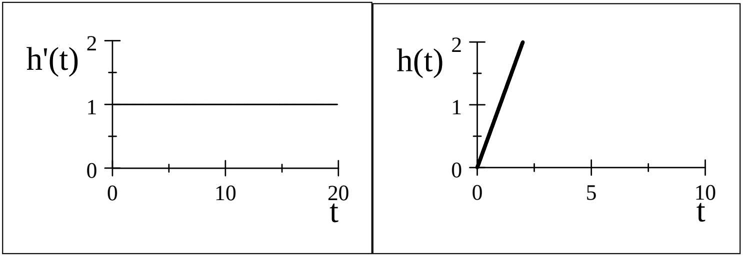
Differentiating both sides relative to t

$$V'(t) = \pi r_0^2 h'(t)$$

and considering that $V'(t) = v$, we have: $h'(t) = \dfrac{v}{\pi r_0^2} \longmapsto h(t) = \dfrac{v}{\pi r_0^2} t$ (given $h(0) = 0$).

- Observe that $h'(t)$ is not the same as $V'(t)$.

Letting $v = \pi r_0^2$ satisfies the initial conditions and produces the following graphs for $h(t)$ and $h'(t)$:



And looking more closely at the inverted frustrum cup ...

Example

Let $r(t)$ be the radius of the surface of coffee. Then

$$r(t) = r_0 + mh(t)$$

with some $m > 0$.

In this case, it appears "natural" to think of $h'(t)$ as a linear function based on the linear dependence of the radius $r(t)$ on the depth $h(t)$ which leads to the conclusion that $h'(t)$ is a *linear function* and $h(t)$ is *quadratic.* But as we shall see, this error in qualitative reasoning fails the test by mathematics ...

Frustrum cup analysis

By the conical frustum volume formula, the volume of coffee in the cup at time t is given by:

$$V(t) = \frac{1}{3}\pi[r_0^2 + r_0 r(t) + r^2(t)]h(t)$$

Instead of differentiating both sides of the above equation relative to t, which would make things more convoluted, we consider that $V'(t) \equiv v$ immediately implies $V(t) = vt$ (with $V(0) = 0$); hence, $h(t)$ is to be found from the *cubic equation:*

$$m^2 h^3(t) + 3mr_0 h^2(t) + 3r_0^2 h(t) - 3vt/\pi = 0.$$

The *general formula for the roots* of such an equation in this case yields $h(t)$ explicitly as:

$$h(t) = -\frac{1}{3m^2}\left[3mr_0 + \sqrt[3]{-27m^3 r_0^3 - 81m^4 vt/\pi}\right].$$

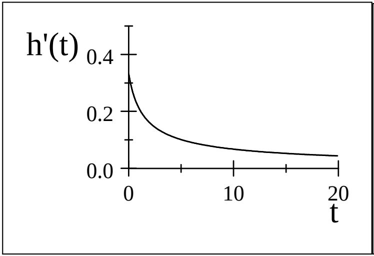
Hence,

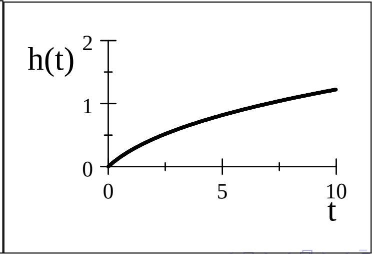
$$h(t) = a(t+b)^{1/3} + c$$

with some $a, b > 0$ and $c < 0$ such that $h(0) = ab^{1/3} + c = 0$ and

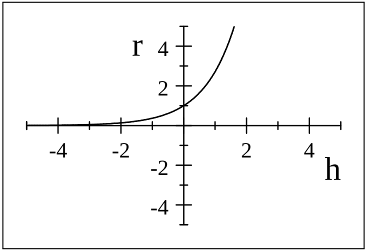
$$h^{'}(t) = \frac{a}{3}(t+b)^{-2/3}.$$

Letting $a = b = 1$ and $c = -1$ satisfies the initial conditions and produces the following graphs for $h(t)$ and $h'(t)$:





Now for a cup with Exponential shaped - sides ...



Using the disk-method from 0 to h

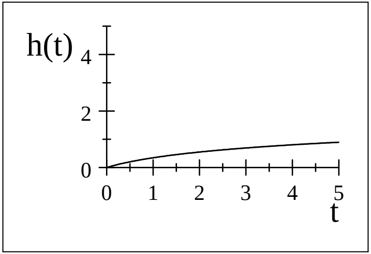
$$V(t) = \pi \int_0^h (e^x)^2 \, dh = \frac{\pi e^{2h}}{2} - \frac{\pi}{2} = vt.$$

Solving for h

$$h(t) = \frac{1}{2} \cdot \ln(\frac{2vt}{\pi} + 1)$$

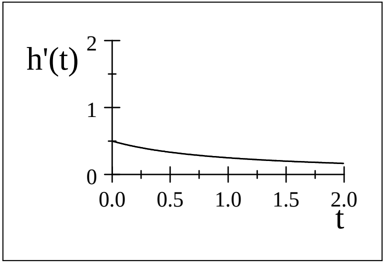
WLOG, let $v = \frac{\pi}{2} ml/\sec$

$$h(t) = \frac{1}{2} \cdot \ln(t+1)$$



Differentiating both sides relative to t

$$h'(t) = \frac{1}{2(t+1)}$$



Reflection

Why must it be the case that the $h'(t)$ graph MUST be concave up?

- Any thoughts?

Anticipated Responses

- In graphing the derivative based on the $h(t)$ graph, the slopes are positive and become less positive tending to 0.
- If $h'(t)$ had a concave down graph, the height would stop at some point, and go backwards.
- Also if concave down, this would produce non-sensical anti-derivative graphs (more on this later).

Taking the Question Deeper

- One evening riding home on my bike I began thinking about this question and what came to mind was ...

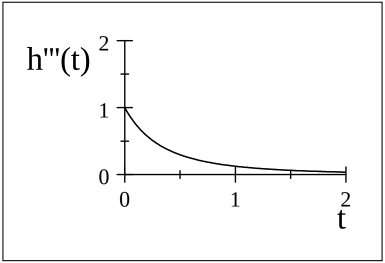

The JERK

The concavity of the $h'(t)$ graph being concave up will of course imply the 'jerk' function $h'''(t) > 0$.
Going back to the exponential-sided cup example ...

$h''(t) = \frac{-1}{2(t+1)^2} < 0$ which makes sense because the graph is decreasing ...

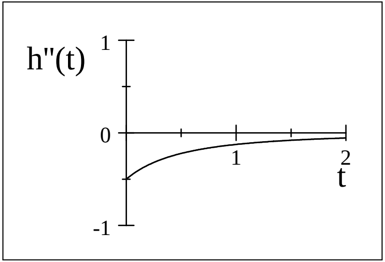
and

$$h'''(t) = \frac{1}{(t+1)^3} > 0$$



Conceptually what can it mean that the jerk must be postive?

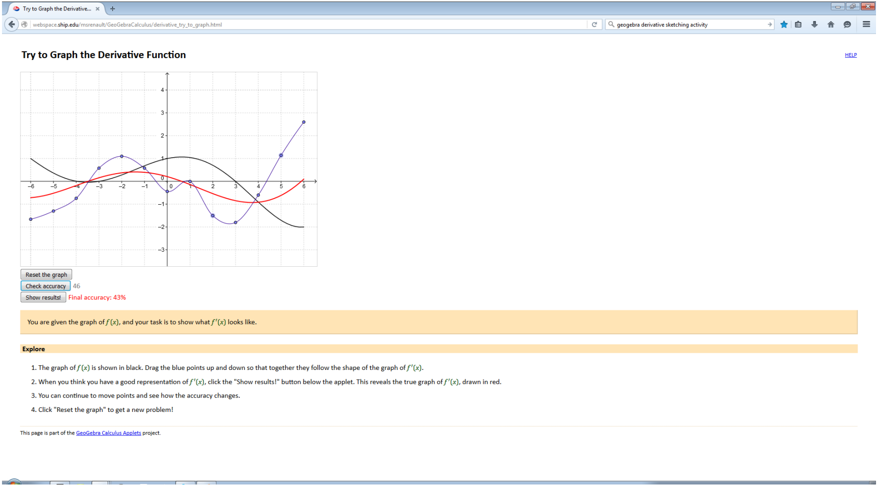
Notice that the rate at which the rate of height change, $h''(t) = \frac{-1}{2(t+1)^2}$ is a deacceleration which is getting 'less' negative.



Luckily for Fred, this 'Jerk' could be thought of as a gradual 'easing off' of the brakes!

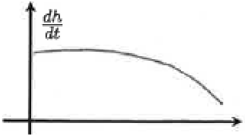
So how do we get students in a position to make qualitatively correct $h'(t)$ graphs?

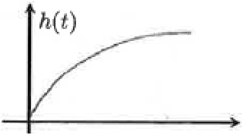
- This semester Cups will be done in Week 15 instead of Week 9
- Changes being made as to how *Tactivities* are assessed
  $\rightarrow$ *photos* – *reflections* – *prompts*–e-portfolios!
- Required for this assignment now are three screen shots added to the portfolio from a Geogebra derivative sketching activity shared with FLOCK by Dr. Tuska.


Try to Graph the Derivative...
webspace.ship.edu/msrenault/GeoGebraCalculus/derivative_try_to_graph.html
geogebra derivative sketching activity
Try to Graph the Derivative Function
HELP
Reset the graph
Check accuracy
46
Show results!
Final accuracy: 43%
You are given the graph of $f(x)$, and your task is to show what $f'(x)$ looks like.
Explore
1. The graph of $f(x)$ is shown in black. Drag the blue points up and down so that together they follow the shape of the graph of $f'(x)$.
2. When you think you have a good representation of $f'(x)$, click the "Show results!" button below the applet. This reveals the true graph of $f'(x)$, drawn in red.
3. You can continue to move points and see how the accuracy changes.
4. Click "Reset the graph" to get a new problem!
This page is part of the GeoGebra Calculus Applets project.

Example

What are some familiar functions you know that look like your $h(t)$ graphs? After discussion with your table, write a reflection paragraph on their derivative graphs in the context of the problem.





- $y = \sqrt{x}$

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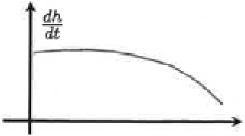
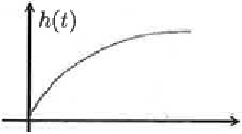
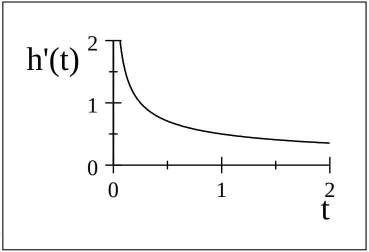
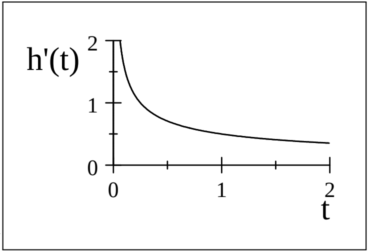




- $y = \sqrt{x}$
- $y = \ln(x+1)$

- $h'(t) = \frac{1}{2\sqrt{t}} \rightarrow$ 

... what about this graph?

- $h'(t) = \frac{1}{2\sqrt{t}} \rightarrow$ 

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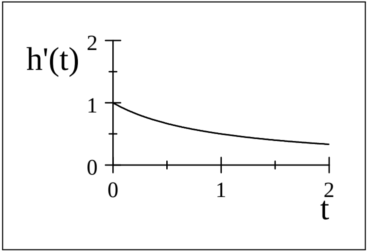
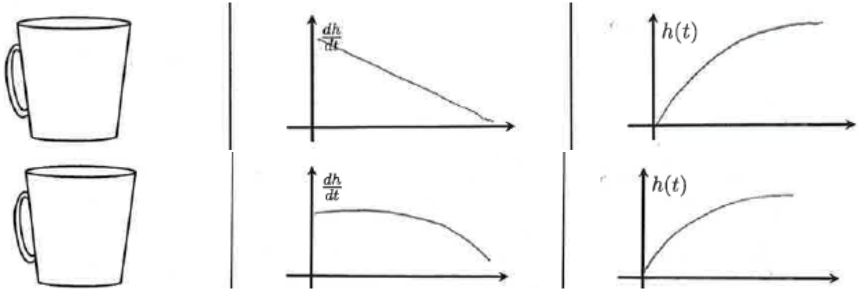
- $h'(t) = \frac{1}{t+1} \rightarrow$ 

... this is better. Why?

Example

What are some familiar functions you know that look like your $h'(t)$ graphs? After discussion with your table, write a reflection paragraph on their anti-derivative graphs in the context of the problem.

(be sure your anti-derivative satisfies $h(0) = 0$)!



- $h'(t) = -2t + 3$

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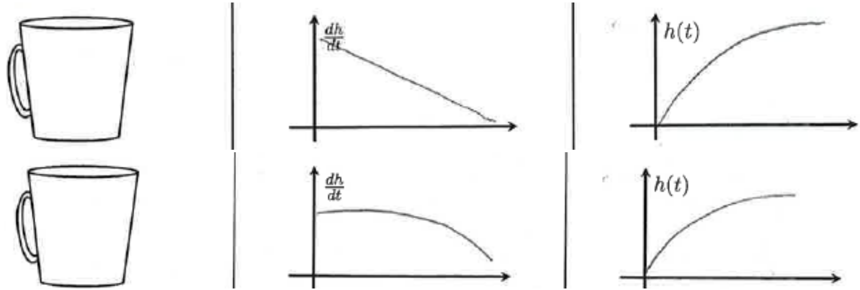
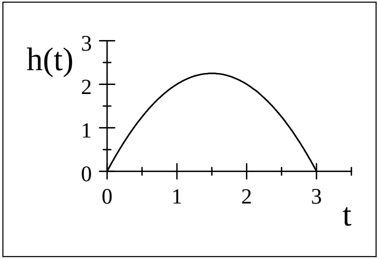
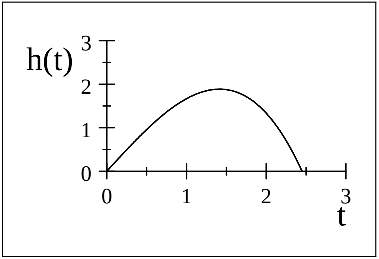


- $h'(t) = -2t + 3$
- $h'(t) = -t^2 + 2$

$$\int -2t + 3 \; dt \rightarrow -t^2 + 3t + C \rightarrow C = 0$$
$$\rightarrow h(t) = -t^2 + 3t$$



$\int -t^2 + 2\ dt \rightarrow -\frac{t^3}{3} + 2t + C \rightarrow C = 0$
$\rightarrow h(t) = -\frac{t^3}{3} + 2t$



Gestalt Therapy - a novel approach for Active Learning 'Therapy' in mathematics?

Gestalt therapy focuses more on process (what is happening) than content (what is being discussed). The emphasis is on what is being done, thought and felt at the moment rather than on what was, might be, could be, or should be.

What is essential in developing a mathematical Wholecept is not studying the component parts themselves, but rather the relationships of the component parts to each other.

How might Gestalt Theory Inform Active Learning of Mathematics?

From (2004) Laura Wagner-Moore, Gestalt Therapy : Past, Present, Theory and Research, Psychotherapy: Theory Research, Practice, Training; Vol. 41, No. 2, 180-189.

- Figure-Background gestalt formation and destruction.
- A mass of unstructured individual data in the environment are subjectively structured by the person into wholes that have form and structure.
- The person's experience and understanding is determined by the Gestalt rather than the raw pieces of data.

From (2004) Laura Wagner-Moore, Gestalt Therapy : Past, Present, Theory and Research, Psychotherapy: Theory Research, Practice, Training; Vol. 41, No. 2, 180-189.

- If a need arises it becomes foreground; if it is satisfied it becomes background and the gestalt is completed *[closure]*.
- Pathology arises when this process is disrupted and unmet needs form incomplete Gestalten interfering with the formation of new ones.*[mental blocks].*

- A need may be blocked by an unclear sensation or lack of awareness of one's needs (Greenberg & Rice, 1977).
- Awareness and *awareness of awareness* of one's experience and needs is considered the 'royal road to the cure.' (Greenberg & Rice, 1977).

- Gestalt Therapy is not about 'curing' or 'talking' about problems.
- It is using an active relationship and active methods to help the patient gain the self-support necessary to *solve problems*.
- In my future work, I aim to explore the methods of Gestalt Therapy and apply them to the teaching of mathematics to help students gain the 'interior' self-support they need to better understand and solve mathematics problems.

Thank you for coming to my talk!

- And in particular I would like to thank my FLOCK colleagues for the many valuable interactions and collaborations you have provided me, and in particular, Dr. Marat Markin, for his collaboration on this talk and future paper.