Math 139 – The Graph of a Rational Function
3 examples

General Steps to Graph a Rational Function

1) Factor the numerator and the denominator

2) State the domain and the location of any holes in the graph

3) Simplify the function by factoring if possible.

4) Find the y-intercept ($x = 0$) and the x-intercept(s) ($y = 0$)

5) Identify any existing asymptotes (vertical, horizontal, or oblique/slant)

General Steps to Graph a Rational Function contd.

6) Identify any points intersecting a horizontal or oblique asymptote.

7) Use test points between the zeros and vertical asymptotes to locate the graph above or below the x-axis

8) Analyze the behavior of the graph on each side of an asymptote

9) Sketch the graph

Example #1

$$f(x) = \frac{x^2 + x - 12}{x^2 - 4}$$

1) Factor the numerator and the denominator

$$f(x) = \frac{(x+4)(x-3)}{(x+2)(x-2)}$$

2) State the domain and the location of any holes in the graph

Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

No holes (They occur where factors cancel)

3) Simplify the function to lowest terms

$$f(x) = \frac{(x+4)(x-3)}{(x+2)(x-2)}$$

4) Find the y-intercept (x = 0) and the x-intercept(s) (y = 0)

y-intercept (x = 0)

$$f(0) = \frac{(0+4)(0-3)}{(0+2)(0-2)}$$

$$f(0) = \frac{-12}{-4} = 3$$

$$(0, 3)$$

x-intercept(s) (y = 0)

These are the roots of the numerator.

$$x + 4 = 0 \qquad x - 3 = 0$$

$$x = -4 \qquad x = 3$$

$$(-4, 0) \qquad (3, 0)$$

5) Identify any existing asymptotes (vertical, horizontal, or oblique

$$f(x) = \frac{x^2 + x - 12}{x^2 - 4}$$

$$f(x) = \frac{(x+4)(x-3)}{(x+2)(x-2)}$$

Horiz. Or Oblique Asymptotes

Examine the largest exponents

Same ∴ Horiz. - use coefficients

$$y = \frac{1}{1}$$

$$HA\text{: } y = 1$$

Vertical Asymptotes

Use denominator factors

$$x + 2 = 0 \qquad x - 2 = 0$$

$$x = -2 \qquad x = 2$$

$$VA\text{: } x = -2 \text{ and } x = 2$$

Oblique Asymptotes occur when degree is 1 greater on top.

Divide using base-x and get rid of denominator to find line.

6) Identify any points intersecting a horizontal or oblique asymptote. (this is possible because unlike a vertical asymptote, the function is defined at these types)

$$y = 1 \quad and \quad f(x) = \frac{x^2 + x - 12}{x^2 - 4}$$

$$1 = \frac{x^2 + x - 12}{x^2 - 4}$$

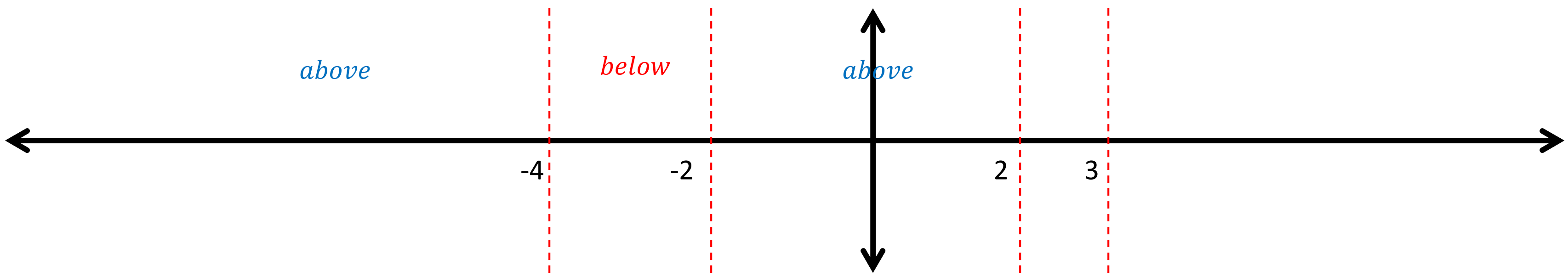
$$x^2 - 4 = x^2 + x - 12$$

$$-4 = x - 12$$

$$8 = x$$

$$(8,1)$$

7) Use test points between the zeros and vertical asymptotes to locate the graph above or below the x-axis



$$f(x) = \frac{(x+4)(x-3)}{(x+2)(x-2)}$$

$$f(-5) = \frac{(-5+4)(-5-3)}{(-5+2)(-5-2)}$$

$$f(-5) = \frac{(-)(-)}{(-)(-)} = +$$

$$f(-5) = above$$

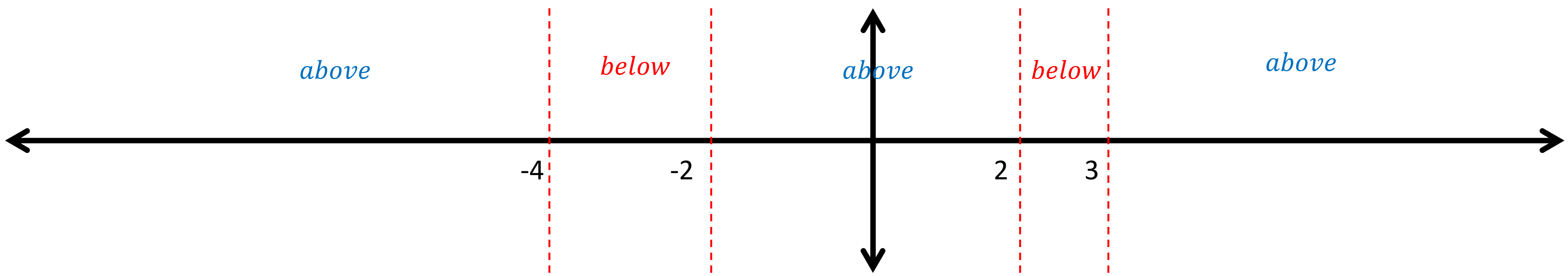
$$f(-3) = \frac{(+)(-)}{(-)(-)} = -$$

$$f(-3) = below$$

$$f(0) = \frac{(+)(-)}{(+)(-)} = +$$

$$f(0) = above$$

7) Use test points between the zeros and vertical asymptotes to locate the graph above or below the x-axis



$$f(x) = \frac{(x+4)(x-3)}{(x+2)(x-2)}$$

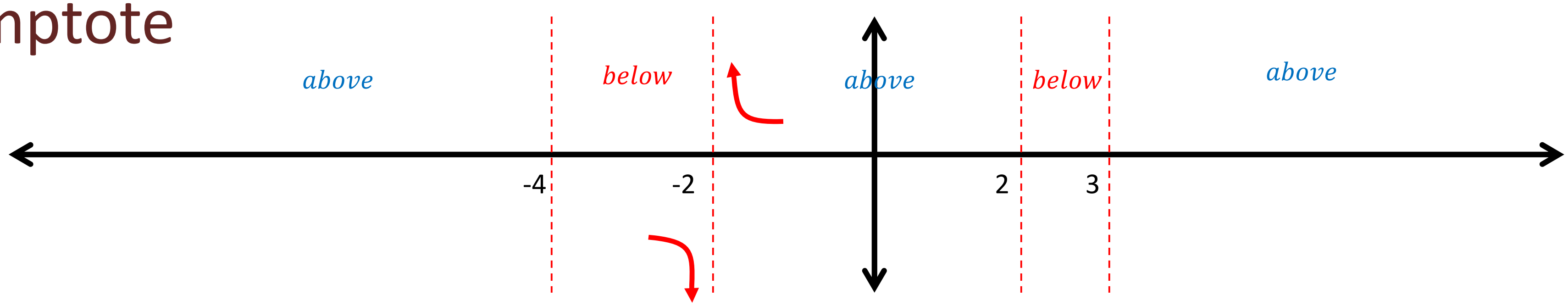
$$f(2.5) = \frac{(+)(-)}{(+)(+)} = -$$

$$f(2.5) = below$$

$$f(4) = \frac{(+)(+)}{(+)(+)} = +$$

$$f(4) = above$$

8) Analyze the behavior of the graph on each side of an asymptote

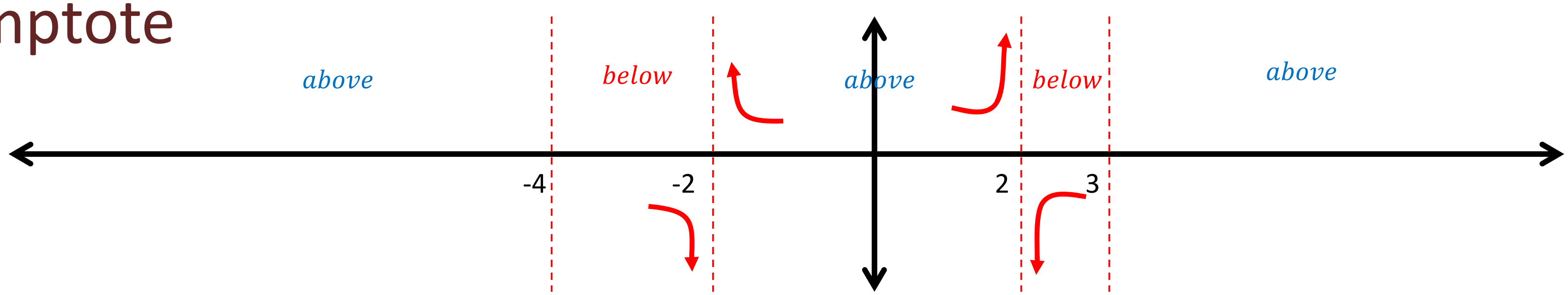


$$f(x) = \frac{(x+4)(x-3)}{(x+2)(x-2)}$$

$$x \to -2^- \quad f(x) \to \frac{(+)(-)}{(0^-)(-)} \quad f(x) \to -\infty$$

$$x \to -2^+ \quad f(x) \to \frac{(+)(-)}{(0^+)(-)} \quad f(x) \to \infty$$

8) Analyze the behavior of the graph on each side of an asymptote

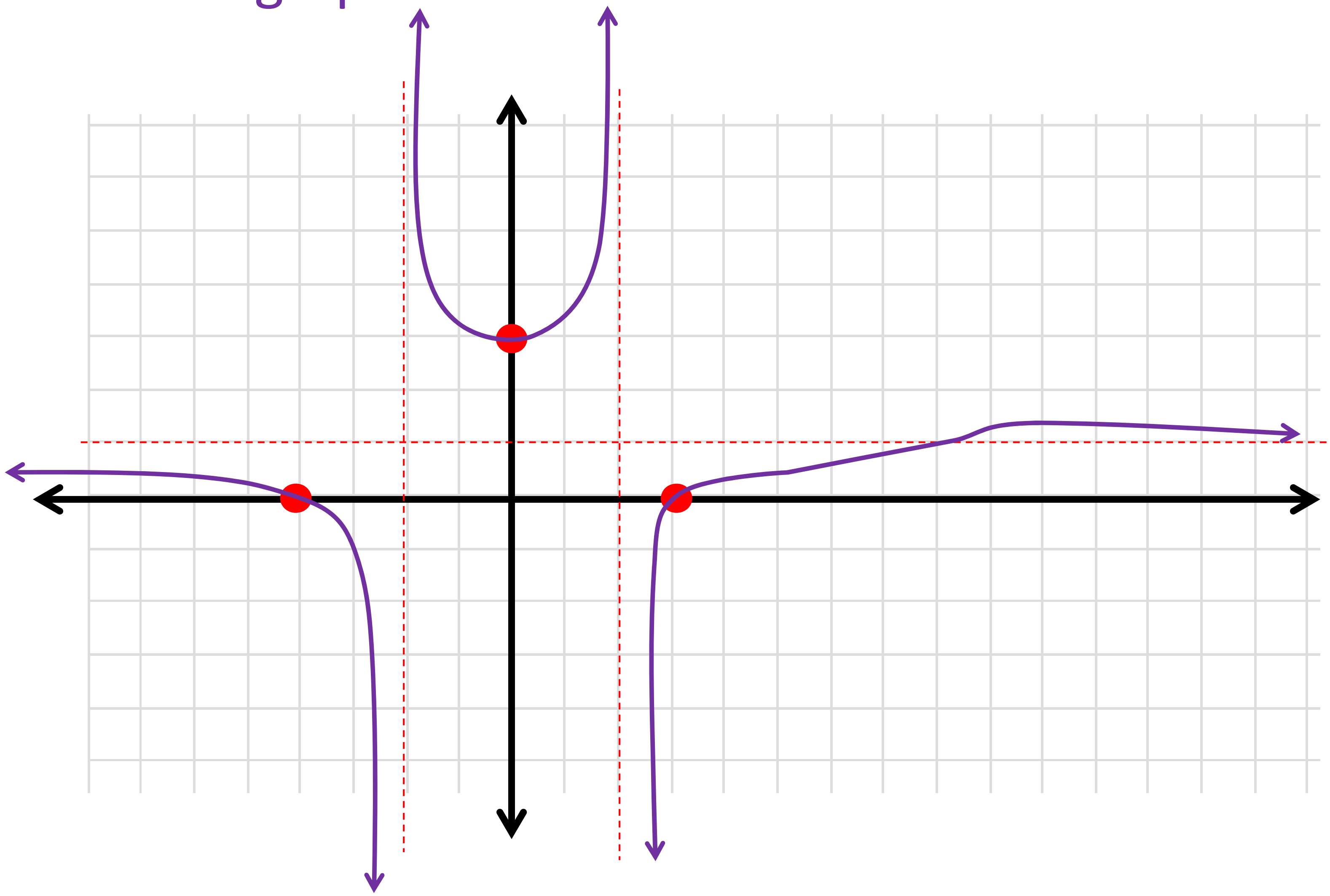


$$f(x) = \frac{(x+4)(x-3)}{(x+2)(x-2)}$$

$$x \to 2^{-} \qquad f(x) \to \frac{(+)(-)}{(+)(0^{-})} \qquad f(x) \to \infty$$

$$x \to 2^{+} \qquad f(x) \to \frac{(+)(-)}{(+)(0^{+})} \qquad f(x) \to -\infty$$

9) Sketch the graph



Example #2

$$f(x) = \frac{x^2 + 3x - 10}{x^2 + 8x + 15}$$

1) Factor the numerator and the denominator

$$f(x) = \frac{(x+5)(x-2)}{(x+5)(x+3)}$$

2) State the domain and the location of any holes in the graph

Domain: $(-\infty, -5) \cup (-5, -3) \cup (-3, \infty)$

Hole in the graph at $x = -5$

3) Simplify the function to lowest terms

$$f(x) = \frac{(x-2)}{(x+3)}$$

4) Find the y-intercept (x = 0) and the x-intercept(s) (y = 0)

y-intercept (x = 0)

$$f(0) = \frac{(0-2)}{(0+3)}$$

$$f(0) = -\frac{2}{3}$$

$$(0, -\frac{2}{3})$$

x-intercept(s) (y = 0)

Use numerator factors

$$x - 2 = 0$$

$$x = 2$$

$$(2, 0)$$

5) Identify any existing asymptotes (vertical, horizontal, or oblique

$$f(x) = \frac{x^2 + 3x - 10}{x^2 + 8x + 15} \qquad f(x) = \frac{(x-2)}{(x+3)}$$

Horiz. Or Oblique Asymptotes

Examine the largest exponents

Same ∴ Horiz. - use coefficients

$$y = \frac{1}{1}$$

$$HA: \; y = 1$$

Vertical Asymptotes

Use denominator factors

$$x + 3 = 0$$

$$x = -3$$

$$VA: x = -3$$

6) Identify any points intersecting a horizontal or oblique asymptote.

$$y = 1 \quad and \quad f(x) = \frac{x-2}{x+3}$$

$$1 = \frac{x-2}{x+3}$$

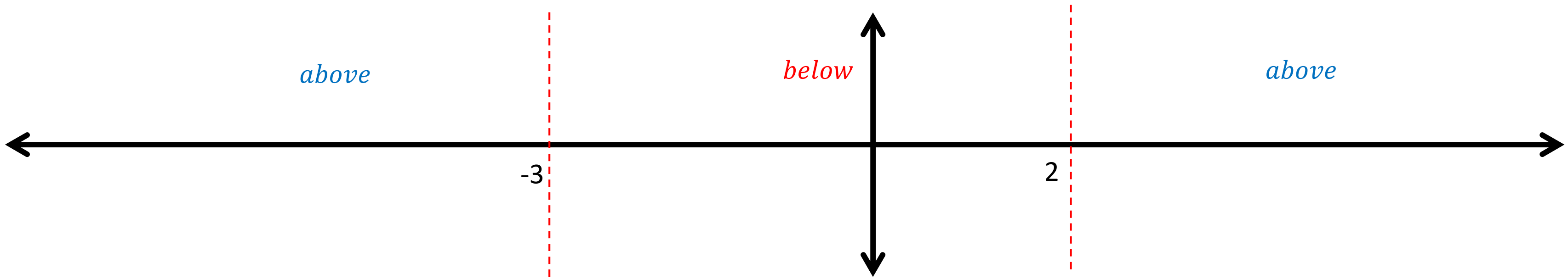
$$x + 3 = x - 2$$

$$3 = -2$$

$$lost\ variable$$

$$no\ points\ of\ intersection\ on\ the\ asymptote$$

7) Use test points between the zeros and vertical asymptotes to locate the graph above or below the x-axis



$$f(x) = \frac{(x-2)}{(x+3)}$$

$$f(-4) = \frac{(-4-2)}{(-4+3)}$$

$$f(-4) = \frac{(-)}{(-)} = +$$

$$f(-4) = above$$

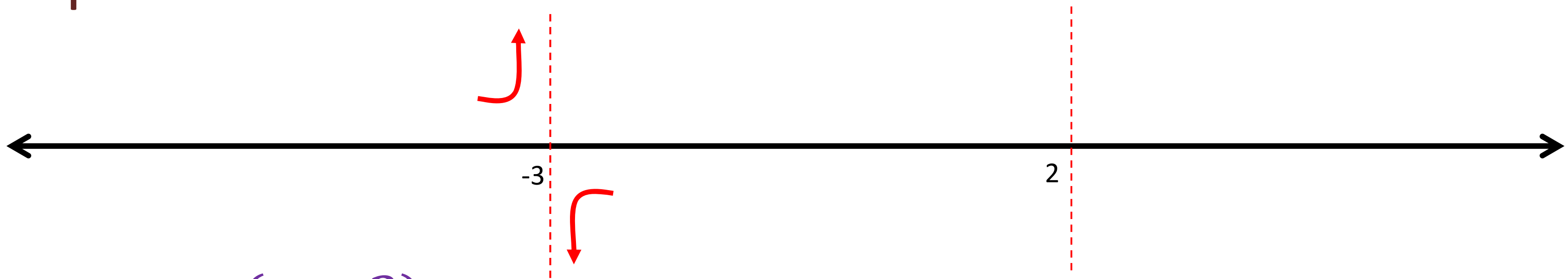
$$f(0) = \frac{(-)}{(+)} = -$$

$$f(0) = below$$

$$f(3) = \frac{(+)}{(+)} = +$$

$$f(3) = above$$

8) Analyze the behavior of the graph on each side of an asymptote

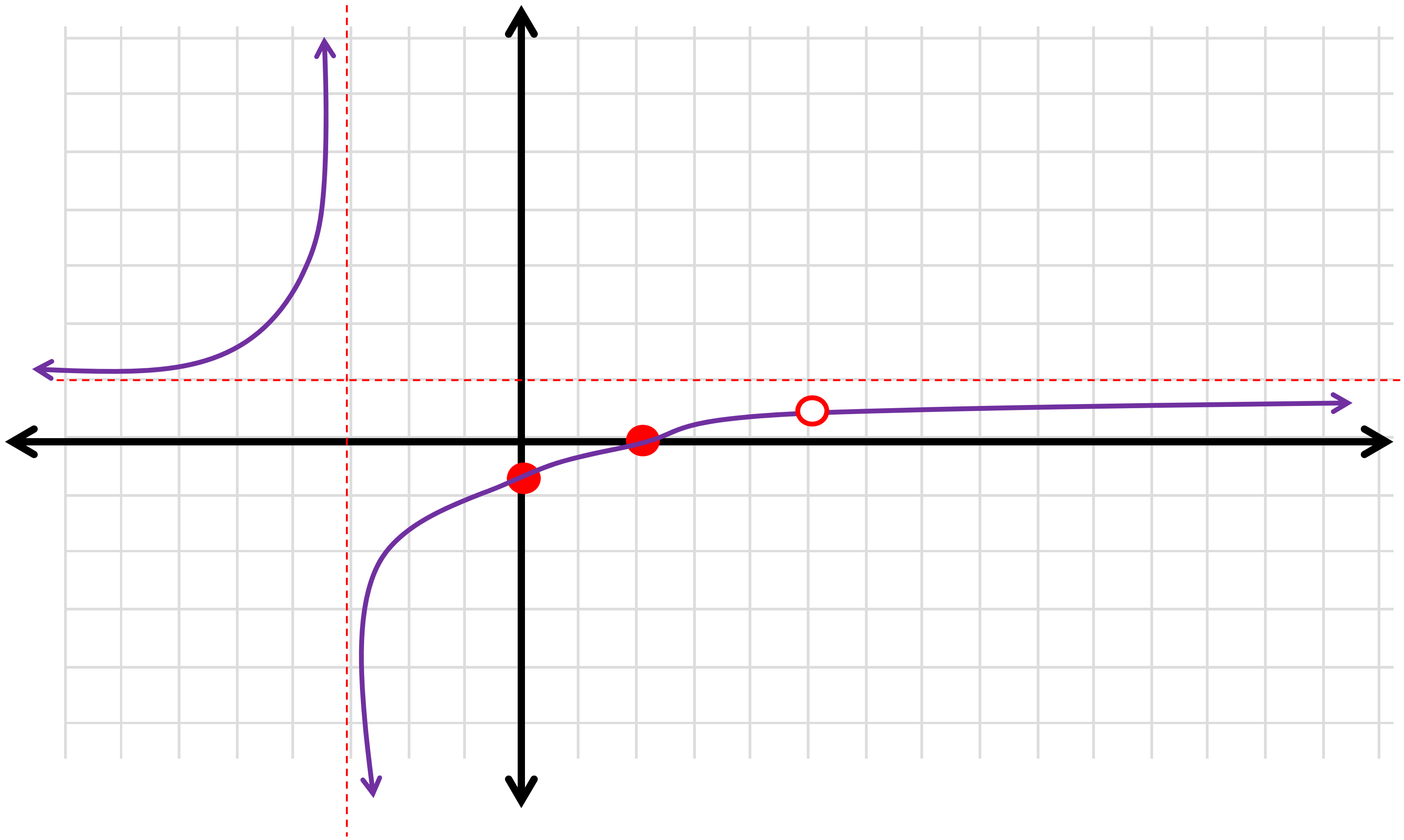


$$f(x) = \frac{(x-2)}{(x+3)}$$

$x \to -3^-$ $\quad f(x) \to \frac{(-)}{(0^-)}$ $\quad f(x) \to \infty$

$x \to -3^+$ $\quad f(x) \to \frac{(-)}{(0^+)}$ $\quad f(x) \to -\infty$

9) Sketch the graph



Example #3

$$f(x) = \frac{x^2 + 3x + 2}{x - 1}$$

1) Factor the numerator and the denominator

$$f(x) = \frac{(x + 2)(x + 1)}{x - 1}$$

2) State the domain and the location of any holes in the graph

Domain: $(-\infty, 1) \cup (1, \infty)$

No holes

3) Simplify the function to lowest terms

$$f(x) = \frac{(x + 2)(x + 1)}{(x - 1)}$$

4) Find the y-intercept (x = 0) and the x-intercept(s) (y = 0)

y-intercept (x = 0)

$$f(0) = \frac{(0+2)(0+1)}{(0-1)}$$

$$f(0) = \frac{2}{-1} = -2$$

$$(0, -2)$$

x-intercept(s) (y = 0)

Use numerator factors

$$x + 2 = 0 \qquad x + 1 = 0$$

$$x = -2 \qquad x = -1$$

$$(-2, 0) \qquad (-1, 0)$$

5) Identify any existing asymptotes (vertical, horizontal, or oblique

$$f(x) = \frac{x^2 + 3x + 2}{x - 1}$$

$$f(x) = \frac{(x+2)(x+1)}{(x-1)}$$

Horiz. or Oblique Asymptotes

Examine the largest exponents

Oblique: Use long division

$$\begin{array}{r} x \quad +4 \\ x - 1 \,\big)\overline{x^2 + 3x + 2} \\ -x^2 - +x \\ \hline 4x \quad +2 \\ -4x - +4 \\ \hline 0 \end{array}$$

$OA:\ y = x + 4$

Vertical Asymptotes

Use denominator factors

$x - 1 = 0$

$x = 1$

$VA: x = 1$

6) Identify any points intersecting a horizontal or oblique asymptote.

$$y = x + 4 \quad and \quad f(x) = \frac{(x+2)(x+1)}{x-1}$$

$$x + 4 = \frac{(x+2)(x+1)}{x-1}$$

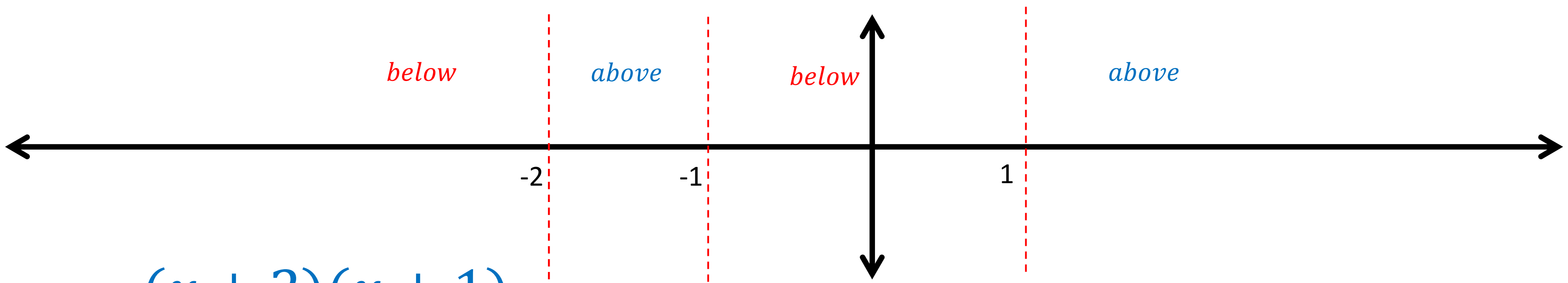
$$(x+4)(x-1) = (x+2)(x+1)$$

$$x^2 + 3x - 4 = x^2 + 3x + 2$$

lost variable

no points of intersection on the asymptote

7) Use test points between the zeros and vertical asymptotes to locate the graph above or below the x-axis



$$f(x) = \frac{(x+2)(x+1)}{(x-1)}$$

$$f(-1.5) = \frac{(+)(-)}{(-)} = +$$

$$f(-4) = \frac{(-)(-)}{(-)} = -$$

$$f(-1.5) = above$$

$$f(3) = \frac{(+)(+)}{(+)} = +$$

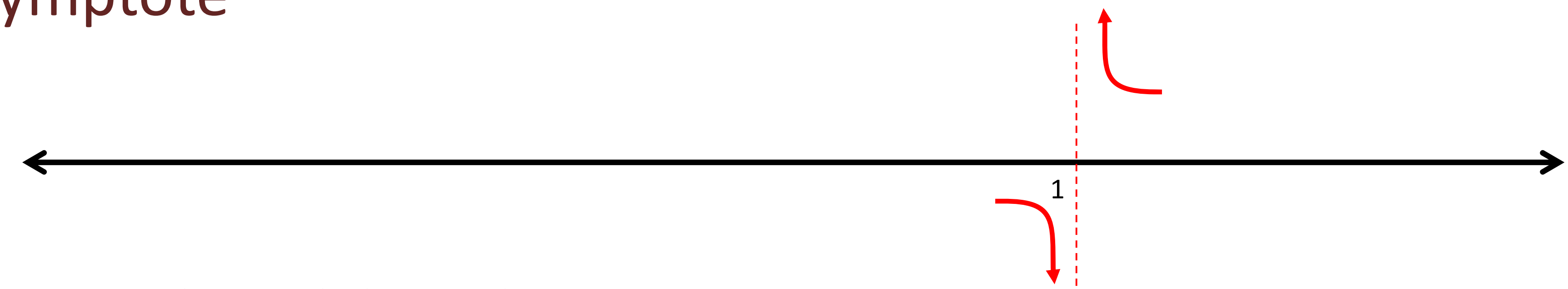
$$f(-4) = below$$

$$f(0) = \frac{(+)(+)}{(-)} = -$$

$$f(3) = above$$

$$f(0) = below$$

8) Analyze the behavior of the graph on each side of an asymptote



$$f(x) = \frac{(x+2)(x+1)}{(x-1)}$$

$x \to 1^-$ $\quad f(x) \to \frac{(+)(+)}{(0^-)}$ $\quad f(x) \to -\infty$

$x \to 1^+$ $\quad f(x) \to \frac{(+)(+)}{(0^+)}$ $\quad f(x) \to \infty$

9) Sketch the graph