Philosophy of Mathematics and Its Logic: Introduction

Stewart Shapiro

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Abstract and Keywords

Mathematics plays an important role in virtually every scientific effort, no matter what part of the world it is aimed at. There is scarcely a natural or a social science that does not have substantial mathematics prerequisites. The burden on any complete philosophy of mathematics is to show how mathematics is applied to the material world, and to show how the methodology of mathematics (whatever it may be) fits into the methodology of the sciences (whatever it may be). In addition to its role in science, mathematics itself seems to be a knowledge-gathering activity. The philosophy of mathematics is, at least in part, a branch of epistemology. However, mathematics is at least prima facie different from other epistemic endeavors.

Keywords: mathematics, philosophy of mathematics, methodology of mathematics, epistemology, material world, logic

1. Motivation, or What We Are Up to

From the beginning, Western philosophy has had a fascination with mathematics. The entrance to Plato's Academy is said to have been marked with the words "Let no one ignorant of geometry enter here." Some major historical mathematicians, such as René Descartes, Gottfried Leibniz, and Blaise Pascal, were also major philosophers. In more recent times, there are Bernard Bolzano, Alfred North Whitehead, David Hilbert, Gottlob Frege, Alonzo Church, Kurt Gödel, and Alfred Tarski. Until very recently, just about every philosopher was aware of the state of mathematics and took it seriously for philosophical attention.

Often, the relationship went beyond fascination. Impressed with the certainty and depth of mathematics, Plato made mathematical ontology the model for his Forms, and mathematical knowledge the model for knowledge generally—to the extent of downplaying or outright neglecting information gleaned from the senses. A similar theme reemerged in the dream of traditional rationalists of extending what (p. 4) they took to be the methodology of mathematics to all scientific and philosophical knowledge. For some rationalists, the goal was to emulate Euclid's *Elements of Geometry*, providing axioms and demonstrations of philosophical principles. Empiricists, the main opponents of rationalism, realized that their orientation to knowledge does not seem to make much sense of mathematics, and they went to some lengths to accommodate mathematics—often distorting it beyond recognition (see Parsons [1983, essay 1]).

Mathematics is a central part of our best efforts at knowledge. It plays an important role in virtually every scientific effort, no matter what part of the world it is aimed at. There is scarcely a natural or a social science that does not have substantial mathematics prerequisites. The burden on any complete philosophy of mathematics is to show how mathematics is applied to the material world, and to show how the methodology of mathematics (whatever it may be) fits into the methodology of the sciences (whatever *it* may be). (See chapter 20 in this volume.)

In addition to its role in science, mathematics itself seems to be a knowledge-gathering activity. We speak of what theorems a given person knows and does not know. Thus, the philosophy of mathematics is, at least in part, a branch of epistemology. However, mathematics is at least prima facie different from other epistemic endeavors. Basic mathematical principles, such as “7 + 5 = 12” or “there are infinitely many prime numbers,” are sometimes held up as paradigms of necessary truths and, a priori, infallible knowledge. It is beyond question that these propositions enjoy a high degree of certainty—however this certainty is to be expounded. How can these propositions be false? How can any rational being doubt them? Indeed, mathematics seems essential to any sort of reasoning at all. Suppose, in the manner of Descartes's first Meditation, that one manages to doubt, or pretend to doubt, the basic principles of mathematics. Can he go on to think at all?

In these respects, at least, logic is like mathematics. At least some of the basic principles of logic are, or seem to be, absolutely necessary and a priori knowable. If one doubts the basic principles of logic, then, perhaps by definition, she cannot go on to think coherently at all. Prima facie, to think coherently just is to think logically.

Like mathematics, logic has also been a central focus of philosophy, almost from the very beginning. Aristotle is still listed among the four or five most influential logicians ever, and logic received attention throughout the ancient and medieval intellectual worlds. Today, of course, logic is a thriving branch of both mathematics and philosophy.

It is incumbent on any complete philosophy of mathematics and any complete philosophy of logic to account for their at least apparent necessity and apriority. Broadly speaking, there are two options. The straightforward way to show that a given discipline appears a certain way is to demonstrate that it is that way. Thus the philosopher can articulate the notions of necessity and apriority, and then show how they apply to mathematics and/or logic. Alternatively, the philosopher can (p. 5) argue that mathematics and/or logic does not enjoy these properties. On this option, however, the philosopher still needs to show why it *appears* that mathematics and/or logic is necessary and a priori. She cannot simply ignore the long-standing belief concerning the special status of these disciplines. There must be something about mathematics and/or logic that has led so many to hold, perhaps mistakenly, that they are necessary and a priori knowable.

The conflict between rationalism and empiricism reflects some tension in the traditional views concerning mathematics, if not logic. Mathematics seems necessary and a priori, and yet it has something to do with the physical world. How is this possible? How can we learn something important about the physical world by a priori reflection in our comfortable armchairs? As noted above, mathematics is essential to any scientific understanding of the world, and science is empirical if anything is—rationalism notwithstanding. Immanuel Kant's thesis that arithmetic and geometry are synthetic a priori was a heroic attempt to reconcile these features of mathematics. According to Kant, mathematics relates to the forms of ordinary perception in space and time. On this view, mathematics applies to the physical world because it concerns the ways that we perceive the physical world. Mathematics concerns the underlying structure and presuppositions of the natural sciences. This is how mathematics gets “applied.” It is necessary because we cannot structure the physical world in any other way. Mathematical knowledge is a priori because we can uncover these presuppositions without any particular experience (chapter 2 of this volume). This set the stage for over two centuries of fruitful philosophy.

2. Global Matters

For any field of study X, the main purposes of the philosophy of X are to interpret X and to illuminate the place of X in the overall intellectual enterprise. The philosopher of mathematics immediately encounters sweeping issues, typically concerning all of mathematics. Most of these questions come from general philosophy: matters of ontology, epistemology, and logic. What, if anything, is mathematics about? How is mathematics pursued? Do we know mathematics and, if so, how do we know mathematics? What is the methodology of mathematics, and to what extent is this methodology reliable? What is the proper logic for mathematics? To what extent are the principles of mathematics objective and independent of the mind, language, and social structure of mathematicians? Some problems and issues on the agenda of contemporary philosophy have remarkably clean formulations when applied to mathematics. Examples include matters of ontology, logic, objectivity, knowledge, and mind.

(p. 6)

The philosopher of logic encounters a similar range of issues, with perhaps less emphasis on ontology. Given the role of deduction in mathematics, the philosophy of mathematics and the philosophy of logic are intertwined, to the point that there is not much use in separating them out.

A mathematician who adopts a philosophy of mathematics should gain something by this: an orientation toward the work, some insight into the role of mathematics, and at least a tentative guide to the direction of mathematics—What sorts of problems are important? What questions should be posed? What methodologies are reasonable? What is likely to succeed? And so on?

One global issue concerns whether mathematical *objects*—numbers, points, functions, sets—exist and, if they do, whether they are independent of the mathematician, her mind, her language, and so on. Define *realism in ontology* to be the view that at least some mathematical objects exist objectively. According to ontological realism, mathematical objects are prima facie abstract, acausal, indestructible, eternal, and not part of space and time. Since mathematical objects share these properties with Platonic Forms, realism in ontology is sometimes called “Platonism.”

Realism in ontology does account for, or at least recapitulate, the necessity of mathematics. If the subject matter of mathematics is as these realists say it is, then the truths of mathematics are independent of anything contingent about the physical universe and anything contingent about the human mind, the community of mathematicians, and so on. What of apriority? The connection with Plato might suggest the existence of a quasi-mystical connection between humans and the abstract and detached mathematical realm. However, such a connection is denied by most contemporary philosophers. As a philosophy of mathematics, “platonism” is often written with a lowercase ‘p,’ probably to mark some distance from the master on matters of epistemology. Without this quasi-mystical connection to the mathematical realm, the ontological realist is left with a deep epistemic problem. If mathematical objects are in fact abstract, and thus causally isolated from the mathematician, then how is it possible for this mathematician to gain knowledge of them? It is close to a piece of incorrigible data that we do have at least some mathematical knowledge. If the realist in ontology is correct, how is this possible?

Georg Kreisel is often credited with shifting attention from the existence of mathematical objects to the objectivity of mathematical truth. Define *realism in truth-value* to be the view that mathematical statements have objective truth-values independent of the minds, languages, conventions, and such of mathematicians. The opposition to this view is *anti-realism in truth-value*, the thesis that if mathematical statements have truth-values at all, these truth-values are dependent on the mathematician.

There is a prima facie alliance between realism in truth-value and realism in ontology. Realism in truth-value is an attempt to develop a view that mathematics (p. 7) deals with objective features of the world. Accordingly, mathematics has the objectivity of a science. Mathematical (and everyday) discourse has variables that range over numbers, and numerals are singular terms. Realism in ontology is just the view that this discourse is to be taken at face value. Singular terms denote objects, and thus numerals denote numbers. According to our two realisms, mathematicians mean what they say, and most of what they say is true. In short, realism in ontology is the default or the first guess of the realist in truth-value.

Nevertheless, a survey of the recent literature reveals that there is no consensus on the logical connections between the two realist theses or their negations. Each of the four possible positions is articulated and defended by established philosophers of mathematics. There are thorough realists (Gödel [1944, 1964], Crispin Wright [1983] and chapter 6 in this volume, Penelope Maddy [1990], Michael Resnik [1997], Shapiro [1997]); thorough anti-realists (Michael Dummett [1973, 1977]) realists in truth-value who are anti-realists in ontology (Geoffrey Hellman [1989] and chapter 17 in this volume, Charles Chihara [1990] and chapter 15 in this volume); and realists in ontology who are anti-realists in truth-value (Neil Tennant [1987, 1997]).

A closely related matter concerns the relationship between philosophy of mathematics and the practice of mathematics. In recent history, there have been disputes concerning some principles and inferences within mathematics. One example is the law of excluded middle, the principle that for every sentence, either it or its negation is true. In symbols: $A \vee \neg A$. For a second example, a definition is *impredicative* if it refers to a class that contains the object being defined. The usual definition of “the least upper bound” is impredicative because it defines a particular upper bound by referring to the set of all upper bounds. Such principles have been criticized on philosophical grounds, typically by anti-realists in ontology. For example, if mathematical objects are mental constructions or creations, then impredicative definitions are circular. One cannot create or construct an object by referring to a class of objects that already contains the item being created or constructed. Realists defended the principles. On that view, a definition does not represent a recipe for creating or constructing a mathematical object. Rather, a definition is a characterization or description of an object that already exists. For a realist in ontology, there is nothing illicit in definitions that refer to classes containing the item in question (see Gödel [1944]). Characterizing “the least upper bound” of a set is no different from defining the “elder poop” to be “the oldest member of the Faculty.”

As far as contemporary mathematics is concerned, the aforementioned disputes are over, for the most part. The law of excluded middle and impredicative definitions are central items in the mathematician's toolbox—to the extent that many practitioners are not aware when these items have been invoked. But this battle was not fought and won on philosophical grounds. Mathematicians did not temporarily don philosophical hats and decide that numbers, say, really do (p. 8) exist independent of the mathematician and, for that reason, decide that it is acceptable to engage in the once disputed methodologies. If anything, the dialectic went in the opposite direction, from mathematics to philosophy. The practices in question were found to be conducive to the practice of mathematics, as mathematics—and thus to the sciences (but see chapters 9, 10, and 19 in this volume).

There is nevertheless a rich and growing research program to see just how much mathematics can be obtained if the restrictions are enforced (chapter 19 in this volume). The research is valuable in its own right, as a study of the logical power of the various once questionable principles. The results are also used to support the underlying philosophies of mathematics and logic.

3. Local Matters

The issues and questions mentioned above concern all of mathematics and, in some cases, all of science. The contemporary philosopher of mathematics has some more narrow foci as well. One group of issues concerns attempts to interpret specific mathematical or scientific results. Many examples come from mathematical logic, and engage issues in the philosophy of logic. The compactness theorem and the Löwenheim–Skolem theorems entail that if a first-order theory has an infinite model at all, then it has a model of every infinite cardinality. Thus, there are unintended, denumerable models of set theory and real analysis. This is despite the fact that we can prove in set theory that the “universe” is uncountable. Arithmetic, the theory of the natural numbers, has uncountable models—despite the fact that by definition a set is countable if and only if it is not larger than the set of natural numbers. What, if anything, do these results say about the human ability to characterize and communicate various concepts, such as notions of cardinality? Skolem (e.g., [1922, 1941]) himself took the results to confirm his view that virtually all mathematical notions are “relative” in some sense. No set is countable or finite *simpliciter*, but only countable or finite relative to some domain or model. Hilary Putnam [1980] espouses a similar relativity. Other philosophers resist the relativity, sometimes by insisting that first-order model theory does not capture the semantics of informal mathematical discourse. This issue may have ramifications concerning the proper logic for mathematics. Perhaps the limitative theorems are an artifact of an incorrect logic (chapters 25 and 26 in this volume).

The wealth of independence results in set theory provide another batch of issues for the philosopher. It turns out that many interesting and important mathematical questions are independent of the basic assertions of set theory. One example is Cantor's *continuum hypothesis* that there are no sets that are strictly (p. 9) larger than the set of natural numbers and strictly smaller than the set of real numbers. Neither the continuum hypothesis nor its negation can be proved in the standard axiomatizations of set theory. What does this independence say about mathematical concepts? Do we have another sort of relativity on offer? Can we only say that a given set is the size of a certain cardinality relative to an interpretation of set theory? Some philosophers hold that these results indicate an indeterminacy concerning mathematical *truth*. There is no fact of the matter concerning, say, the continuum hypothesis. These philosophers are thus anti-realists in truth-value. The issue here has ramifications concerning the practice of mathematics. If one holds that the continuum hypothesis has a determinate truth-value, he or she may devote effort to determining this truth-value. If, instead, someone holds that the continuum hypothesis does not have a determinate truth-value, then he is free to adopt it or not, based on what makes for the most convenient set theory. It is not clear whether the criteria that the realist might adopt to decide the continuum hypothesis are different from the criteria the anti-realist would use for determining what makes for the most convenient theory.

A third example is Gödel's incompleteness theorem that the set of arithmetic truths is not effective. Some take this result to refute mechanism, the thesis that the human mind operates like a machine. Gödel himself held that either the mind is not a machine or there are arithmetic questions that are “absolutely undecidable,” questions that are unanswerable by us humans (see Gödel [1951], Shapiro [1998]). On the other hand, Webb [1980] takes the incompleteness results to support mechanism.

To some extent, some questions concerning the applications of mathematics are among this group of issues. What can a theorem of mathematics tell us about the natural world studied in science? To what extent can we *prove* things about knots, bridge stability, chess endgames, and economic trends? There are (or were) philosophers who take mathematics to be no more than a meaningless game played with symbols (chapter 8 in this volume), but everyone else holds that mathematics has some sort of meaning. What is this meaning, and how does it relate to the meaning of ordinary nonmathematical discourse? What can a theorem tell us about the physical world, about human knowability, about the abilities-in-principle of programmed computers, and so on?

Another group of issues consists of attempts to articulate and interpret particular mathematical *theories* and *concepts*. One example is the foundational work in arithmetic and analysis. Sometimes, this sort of activity has ramifications for mathematics itself, and thus challenges and blurs the boundary between mathematics and its philosophy. Interesting and powerful research techniques are often suggested by foundational work that forges connections between mathematical fields. In addition to mathematical logic, consider the embedding of the natural numbers in the complex plane, via analytic number theory. Foundational activity has spawned whole branches of mathematics.

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Sometimes developments within mathematics lead to unclarity concerning what a certain concept is. The example developed in Lakatos [1976] is a case in point. A series of "proofs and refutations" left interesting and important questions over what a polyhedron is. For another example, work leading to the foundations of analysis led mathematicians to focus on just what a function is, ultimately yielding the modern notion of function as arbitrary correspondence. The questions are at least partly ontological.

This group of issues underscores the *interpretive* nature of philosophy of mathematics. We need to figure out what a given mathematical concept *is*, and what a stretch of mathematical discourse *says*. The Lakatos study, for example, begins with a "proof" consisting of a thought experiment in which one removes a face of a given polyhedron, stretches the remainder out on a flat surface, and then draws lines, cuts, and removes the various parts—keeping certain tallies along the way. It is not clear a priori how this blatantly dynamic discourse is to be understood. What is the logical form of the discourse and what is its logic? What is its ontology? Much of the subsequent mathematical/philosophical work addresses just these questions.

Similarly, can one tell from surface grammar alone that an expression like "dx" is not a singular term denoting a mathematical object, while in some circumstances, "dy/dx" does denote something—but the denoted item is a function, not a quotient? The history of analysis shows a long and tortuous task of showing just what expressions like this mean.

Of course, mathematics can often go on quite well without this interpretive work, and sometimes the interpretive work is premature and is a distraction at best. Berkeley's famous, penetrating critique of analysis was largely ignored among mathematicians—so long as they knew "how to go on," as Ludwig Wittgenstein might put it. In the present context, the question is whether the mathematician must stop mathematics until he has a semantics for his discourse fully worked out. Surely not. On occasion, however, tensions within mathematics lead to the interpretive philosophical/semantic enterprise. Sometimes, the mathematician is not sure how to "go on as before," nor is he sure just what the concepts are. Moreover, we are never certain that the interpretive project is accurate and complete, and that other problems are not lurking ahead.

4. A Potpourri of Positions

I now present sketches of some main positions in the philosophy of mathematics. The list is not exhaustive, nor does the coverage do justice to the subtle and deep work of proponents of each view. Nevertheless, I hope it serves as a useful (p. 11) guide to both the chapters that follow and to at least some of the literature in contemporary philosophy of mathematics. Of course, the reader should not hold the advocates of the views to the particular articulation that I give here, especially if the articulation sounds too implausible to be advocated by any sane thinker.

4.1. Logicism: a Matter of Meaning

According to Alberto Coffa [1991], a major item on the agenda of Western philosophy throughout the nineteenth century was to account for the (at least) apparent necessity and a priori nature of mathematics and logic, and to account for the applications of mathematics, without invoking anything like Kantian intuition. According to Coffa, the most fruitful development on this was the "semantic tradition," running through the work of Bolzano, Frege, the early Wittgenstein, and culminating with the Vienna Circle. The main theme—or insight, if you will—was to locate the source of necessity and a priori knowledge in the use of *language*. Philosophers thus turned their attention to linguistic matters concerning the pursuit of mathematics. What do mathematical assertions mean? What is their logical form? What is the best semantics for mathematical language? The members of the semantic tradition developed and honed many of the tools and concepts still in use today in mathematical logic, and in Western philosophy generally. Michael Dummett calls this trend in the history of philosophy the *linguistic turn*.

An important program of the semantic tradition was to show that at least some basic principles of mathematics are *analytic*, in the sense that the propositions are true in virtue of meaning. Once we understood terms like “natural number,” “successor function,” “addition,” and “multiplication,” we would thereby see that the basic principles of arithmetic, such as the Peano postulates, are true. If the program could be carried out, it would show that mathematical truth is necessary—to the extent that analytic truth, so construed, is necessary. Given what the words mean, mathematical propositions have to be true, independent of any contingencies in the material world. And mathematical knowledge is a priori—to the extent that knowledge of meanings is a priori. Presumably, speakers of the language know the meanings of words a priori, and thus we know mathematical propositions a priori.

The most articulate version of this program is *logicism*, the view that at least some mathematical propositions are true in virtue of their logical forms (chapter 5 in this volume). According to the logicist, arithmetic truth, for example, is a species of logical truth. The most detailed developments are those of Frege [1884, 1893] and Alfred North Whitehead and Bertrand Russell [1910]. Unlike Russell, Frege was a realist in ontology, in that he took the natural numbers to be objects. Thus, for Frege at least, logic has an ontology—there are “logical objects.”

(p. 12)

In a first attempt to define the general notion of cardinal number, Frege [1884, §63] proposed the following principle, which has become known as “Hume's principle”:

> For any concepts *F*, *G*, the number of *F*'s is identical to the number of *G*'s if and only if *F* and *G* are equinumerous.

Two concepts are equinumerous if they can be put in one-to-one correspondence. Frege showed how to define equinumerosity without invoking natural numbers. His definition is easily cast in what is today recognized as pure second-order logic. If second-order logic is logic (chapter 25 in this volume), then Frege succeeded in reducing Hume's principle, at least, to logic.

Nevertheless, Frege balked at taking Hume's principle as the ultimate foundation for arithmetic because Hume's principle only fixes identities of the form “the number of *F*'s = the number of *G*'s.” The principle does not determine the truth-value of sentences in the form “the number of *F*'s = *t*,” where *t* is an arbitrary singular term. This became known as the Caesar problem. It is not that anyone would confuse a natural number with the Roman general Julius Caesar, but the underlying idea is that we have not succeeded in characterizing the natural numbers as objects unless and until we can determine how and why any given natural number is the same as or different from any object whatsoever. The distinctness of numbers and human beings should be a consequence of the theory, and not just a matter of intuition.

Frege went on to provide explicit definitions of individual natural numbers, and of the concept "natural number," in terms of *extensions* of concepts. The number 2, for example, is the extension (or collection) of all concepts that hold of exactly two elements. The inconsistency in Frege's theory of extensions, as shown by Russell's paradox, marked a tragic end to Frege's logicist program.

Russell and Whitehead [1910] traced the inconsistency in Frege's system to the impredicativity in his theory of extensions (and, for that matter, in Hume's principle). They sought to develop mathematics on a safer, predicative foundation. Their system proved to be too weak, and ad hoc adjustments were made, greatly reducing the attraction of the program. There is a thriving research program under way to see how much mathematics can be recovered on a predicative basis (chapter 19 in this volume).

Variations of Frege's original approach are vigorously pursued today in the work of Crispin Wright, beginning with [1983], and others like Bob Hale [1987] and Neil Tennant ([1987, 1997]) (chapter 6 in this volume). The idea is to bypass the treatment of extensions and to work with (fully impredicative) Hume's principle, or something like it, directly. Hume's principle is consistent with second-order logic if second-order arithmetic is consistent (see Boolos [1987] and Hodes [1984]), so at least the program will not fall apart like Frege's did. But what is the philosophical point? On the *neologicist* approach, Hume's principle is taken to (p. 13) be an explanation of the concept of "number." Advocates of the program argue that even if Hume's principle is not itself analytic—true in virtue of meaning—it can become known a priori, once one has acquired a grasp of the concept of cardinal number. Hume's principle is akin to an implicit definition. Frege's own technical development shows that the Peano postulates can be derived from Hume's principle in a standard, higher-order logic. Indeed, the only essential use that Frege made of extensions was to derive Hume's principle—everything else concerning numbers follows from that. Thus the basic propositions of arithmetic enjoy the same privileged epistemic status had by Hume's principle (assuming that second-order deduction preserves this status). Neologicism is a reconstructive program showing how arithmetic propositions can become known.

The neologicist (and Fregean) development makes essential use of the fact that impredicativity of Hume's principle is impredicative in the sense that the variable F in the locution "the number of F's" is instantiated with concepts that themselves are defined in terms of numbers. Without this feature, the derivation of the Peano axioms from Hume's principle would fail. This impredicativity is consonant with the ontological realism adopted by Frege and his neologicist followers. Indeed, the neologicist holds that the left-hand side of an instance of Hume's principle has the same truth conditions as its right-hand side, but the left-hand side gives the proper logical form. Locutions like “the number of *F*'s” are genuine singular terms denoting numbers.

The neologicist project, as developed thus far, only applies basic arithmetic and the natural numbers. An important item on the agenda is to extend the treatment to cover other areas of mathematics, such as real analysis, functional analysis, geometry, and set theory. The program involves the search for abstraction principles rich enough to characterize more powerful mathematical theories (see, e.g., Hale [2000a, 2000b] and Shapiro [2000a, 2003]).

4.2. Empiricism, Naturalism, and Indispensability

Coffa [1982] provides a brief historical sketch of the semantic tradition, outlining its aims and accomplishments. Its final sentence is “And then came Quine.” Despite the continued pursuit of variants of logicism (chapter 26 in this volume), the standard concepts underlying the program are in a state of ill repute in some quarters, notably much of North America. Many philosophers no longer pay serious attention to notions of meaning, analyticity, and a priori knowledge. To be precise, such notions are not given a primary role in the epistemology of mathematics, or anything else for that matter, by many contemporary philosophers. W. V. O. Quine (e.g., [1951, 1960]) is usually credited with initiating widespread skepticism concerning these erstwhile philosophical staples.

(p. 14)

Quine, of course, does not deny that the truth-value of a given sentence is determined by both the use of language and the way the world is. To know that “Paris is in France,” one must know something about the use of the words “Paris,” “is,” and “France,” and one must know some geography. Quine's view is that the linguistic and factual components of a given sentence cannot be sharply distinguished, and thus there is no determinate notion of a sentence being true solely in virtue of language (analytic), as opposed to a sentence whose truth depends on the way the world is (synthetic).

Then how is mathematics known? Quine is a thoroughgoing empiricist, in the tradition of John Stuart Mill (chapter 3 in this volume). His positive view is that *all* of our beliefs constitute a seamless web answerable to, and only to, sensory stimulation. There is no difference in kind between mundane beliefs about material objects, the far reaches of esoteric science, mathematics, logic, and even so-called truths-by-definition (e.g., “no bachelor is married"). The word "seamless" in Quine's metaphor suggests that everything in the web is logically connected to everything else in the web, at least in principle. Moreover, no part of the web is knowable a priori.

This picture gives rise to a now common argument for realism. Quine and others, such as Putnam [1971], propose a hypothetical-deductive epistemology for mathematics. Their argument begins with the observation that virtually all of science is formulated in mathematical terms. Thus, mathematics is "confirmed" to the extent that science is. Because mathematics is indispensable for science, and science is well confirmed and (approximately) true, mathematics is well confirmed and true as well. This is sometimes called the *indispensability argument*.

Thus, Quine and Putnam are realists in truth-value, holding that some statements of mathematics have objective and nonvacuous truth-values independent of the language, mind, and form of life of the mathematician and scientist (assuming that science enjoys this objectivity). Quine, at least, is also a realist in ontology. He accepts the Fregean (and neologicist) view that "existence" is univocal. There is no ground for distinguishing terms that refer to medium-sized physical objects, terms that refer to microscopic and submicroscopic physical objects, and terms that refer to numbers. According to Quine and Putnam, *all* of the items in our ontology—apples, baseballs, electrons, and numbers—are theoretical posits. We accept the existence of all and only those items that occur in our best accounts of the material universe. Despite the fact that numbers and functions are not located in space and time, we know about numbers and functions the same way we know about physical objects—via the role of terms referring to such entities in mature, well-confirmed theories.

Indispensability arguments are anathema to those, like the logicists, logical positivists, and neologicists, who maintain the traditional views that mathematics is absolutely necessary and/or analytic and/or knowable a priori. On such views, mathematical knowledge cannot be dependent on anything as blatantly (p. 15) empirical and contingent as everyday discourse and natural science. The noble science of mathematics is independent of all of that. From the opposing Quinean perspective, mathematics and logic do not enjoy the necessity traditionally believed to hold of them; and mathematics and logic are not knowable a priori.

Indeed, for Quine, *nothing* is knowable a priori. The thesis is that everything in the web—the mundane beliefs about the physical world, the scientific theories, the mathematics, the logic, the connections of meaning—is up for revision if the "data" become sufficiently recalcitrant. From this perspective, mathematics is of a piece with highly confirmed scientific theories, such as the fundamental laws of gravitation. Mathematics appears to be necessary and a priori knowable (only) because it lies at the "center" of the web of belief, farthest from direct observation. Since mathematics permeates the web of belief, the scientist is least likely to suggest revisions in mathematics in light of recalcitrant “data.” That is to say, because mathematics is invoked in virtually every science, its rejection is extremely unlikely, but the rejection of mathematics cannot be ruled out in principle. No belief is incorrigible. No knowledge is a priori, all knowledge is ultimately based on experience (see Colyvan [2001], and chapter 12 in this volume).

The seamless web is of a piece with Quine's *naturalism*, characterized as “the abandonment of first philosophy” and “the recognition that it is within science itself … that reality is to be identified and described” ([1981, p. 72]). The idea is to see philosophy as continuous with the sciences, not prior to them in any epistemological or foundational sense. If anything, the naturalist holds that science is prior to philosophy. *Naturalized epistemology* is the application of this theme to the study of knowledge. The philosopher sees the human knower as a thoroughly natural being within the physical universe. Any faculty that the philosopher invokes to explain knowledge must involve only natural processes amenable to ordinary scientific scrutiny.

Naturalized epistemology exacerbates the standard epistemic problems with realism in ontology. The challenge is to show how a physical being in a physical universe can come to know about *abstracta* like mathematical objects (see Field [1989, essay 7]). Since abstract objects are causally inert, we do not observe them but, nevertheless, we still (seem to) know something about them. The Quinean meets this challenge with claims about the role of mathematics in science. Articulations of the Quinean picture thus should, but usually do not, provide a careful explanation of the application of mathematics to science, rather than just noting the existence of this applicability (chapter 20 in this volume). This explanation would shed light on the abstract, non-spatiotemporal nature of mathematical objects, and the relationships between such objects and ordinary and scientific material objects. How is it that talk of numbers and functions can shed light on tables, bridge stability, and market stability? Such an analysis would go a long way toward defending the Quinean picture of a web of belief.

(p. 16)

Once again, it is a central tenet of the naturalistically minded philosopher that there is no first philosophy that stands prior to science, ready to either justify or criticize it. Science guides philosophy, not the other way around. There is no agreement among naturalists that the same goes for mathematics. Quine himself accepts mathematics (as true) only to the extent that it is applied in the sciences. In particular, he does not accept the basic assertions of higher set theory because they do not, at present, have any empirical applications. Moreover, he advises mathematicians to conform their practice to his version of naturalism by adopting a weaker and less interesting, but better understood, set theory than the one they prefer to work with.

Mathematicians themselves do not follow the epistemology suggested by the Quinean picture. They do not look for confirmation in science before publishing their results in mathematics journals, or before claiming that their theorems are true. Thus, Quine's picture does not account for mathematics as practiced. Some philosophers, such as Burgess [1983] and Maddy [1990, 1997], apply naturalism to mathematics directly, and thereby declare that mathematics is, and ought to be, insulated from much traditional philosophical inquiry, or any other probes that are not to be resolved by mathematicians qua mathematicians. On such views, philosophy of mathematics—naturalist or otherwise—should not be in the business of either justifying or criticizing mathematics (chapters 13 and 14 in this volume).

4.3. No Mathematical Objects

The most popular way to reject realism in ontology is to flat out deny that mathematics has a subject matter. The *nominalist* argues that there are no numbers, points, functions, sets, and so on. The burden on advocates of such views is to make sense of mathematics and its applications without assuming a mathematical ontology. This is indicated in the title of Burgess and Rosen's study of nominalism, *A Subject with No Object* [1997].

A variation on this theme that played an important role in the history of our subject is *formalism*. An extreme version of this view, which is sometimes called “game formalism,” holds that the essence of mathematics is the following of meaningless rules. Mathematics is likened to the play of a game like chess, where characters written on paper play the role of pieces to be moved. All that matters to the pursuit of mathematics is that the rules have been followed correctly. As far as the philosophical perspective is concerned, the formulas may as well be meaningless.

Opponents of game formalism claim that mathematics is inherently informal and perhaps even nonmechanical. Mathematical language has meaning, and it is a gross distortion to attempt to ignore this. At best, formalism focuses on a small (p. 17) aspect of mathematics, the fact that logical consequence is formal. It deliberately leaves aside what is essential to the enterprise.

A different formalist philosophy of mathematics was presented by Haskell Curry (e.g., [1958]). The program depends on a historical thesis that as a branch of mathematics develops, it becomes more and more rigorous in its methodology, the end result being the codification of the branch in formal deductive systems. Curry claimed that assertions of a mature mathematical theory are to be construed not so much as the results of moves *in* a particular formal deductive system (as a game formalist might say), but rather as assertions *about* a formal system. An assertion at the end of a research paper would be understood in the form "such and such is a theorem in this formal system." For Curry, then, mathematics is an objective science, and it has a subject matter—formal systems. In effect, mathematics is metamathematics. (See chapter 8 in this volume for a more developed account of formalism.)

On the contemporary scene, one prominent version of nominalism is *fictionalism*, as developed, for example, by Hartry Field [1980]. Numbers, points, and sets have the same philosophical status as the entities presented in works of fiction. According to the fictionalist, the number 6 is the same kind of thing as Dr. Watson or Miss Marple.

According to Field, mathematical language should be understood at face value. Its assertions have vacuous truth-values. For example, "all natural numbers are prime" comes out true, since there are no natural numbers. Similarly, "there is a prime number greater than 10" is false, and both Fermat's last theorem and the Goldbach conjecture are true. Of course, Field does not exhort mathematicians to settle their open questions via this vacuity. Unlike Quine, Field has no proposals for changing the methodology of mathematics. His view concerns how the results of mathematics should be interpreted, and the role of these results in the scientific enterprise. For Field, the goal of mathematics is not to assert the true. The only mathematical knowledge that matters is knowledge of logical consequences (see Field [1984]).

Field regards the Quine/Putnam indispensability argument to be the only serious consideration in favor of ontological realism. His overall orientation is thus broadly Quinean—in direct opposition to the long-standing belief that mathematical knowledge is a priori. As we have seen, more traditional philosophers—and most mathematicians—regard indispensability as irrelevant to mathematical knowledge. In contrast, for thinkers like Field, once one has undermined the indispensability argument, there is no longer any serious reason to believe in the existence of mathematical objects.

Call a scientific theory "nominalistic" if it is free of mathematical presuppositions. As Quine and Putnam pointed out, most of the theories developed in scientific practice are not nominalistic, and so begins the indispensability argument. The first aspect of Field's program is to develop nominalistic versions of (p. 18) various scientific theories. Of course, Field does not do this for every prominent scientific theory. To do so, he would have to understand every prominent scientific theory, a task that no human can accomplish anymore. Field gives one example—Newtonian gravitational theory—in some detail, to illustrate a technique that can supposedly be extended to other scientific work.

The second aspect of Field's program is to show that the nominalistic theories are sufficient for attaining the scientific goal of determining truths about the physical universe (i.e., accounting for observations). Let P be a nominalistic scientific theory and let S be a mathematical theory together with some “bridge principles” that connect the mathematical terminology with the physical terminology. Define S to be *conservative* over P if for any sentence Φ in the language of the nominalistic theory, if Φ is a consequence of $P + S$, then Φ is a consequence of P alone. Thus, if the mathematical theory is conservative over the nominalist one, then any physical consequence we get via the mathematics we could get from the nominalistic physics alone. This would show that mathematics is dispensable in principle, even if it is practically necessary. Field shows that standard mathematical theories and bridge principles are conservative over his nominalistic Newtonian theory, at least if the conservativeness is understood in model-theoretic terms: if Φ holds in all models of $P + S$, then Φ holds in all models of P.

The sizable philosophical literature generated by Field [1980] includes arguments that Field's technique does not generalize to more contemporary theories like quantum mechanics (Malament [1982]); arguments that Field's distinction between abstract and concrete does not stand up, or that it does not play the role needed to sustain Field's fictionalism (Resnik [1985]); and arguments that Field's nominalistic theories are not conservative in the philosophically relevant sense (Shapiro [1983]). The collection by Field [1989] contains replies to some of these objections.

Another common anti-realist proposal is to reconstrue mathematical assertions in *modal* terms. The philosopher understands mathematical assertions to be about what is possible, or about what would be the case if objects of a certain sort existed. The main innovation in Chihara [1990] is a modal primitive, a “constructibility quantifier.” If Φ is a formula and x a certain type of variable, then Chihara's system contains a formula that reads “it is possible to construct an x such that Φ.” According to Chihara, constructibility quantifiers do not mark what Quine calls “ontological commitment.” Common sense supports this—to the extent that the notion of ontological commitment is part of common sense. If someone says that it is possible to construct a new ballpark in Boston, she is not asserting the existence of any ballpark, nor is she asserting the existence of a strange entity called a “possible ballpark.” She only speaks of what it is possible to do.

The formal language developed in Chihara [1990] includes variables that range over *open sentences* (i.e., sentences with free variables), and these open-sentence variables can be bound by constructibility quantifiers. With keen attention to detail, (p. 19) Chihara develops arithmetic, analysis, functional analysis, and so on in his system, following the parallel development of these mathematical fields in simple (impredicative) type theory.

Unlike Field, Chihara is a realist in truth-value. He holds that the relevant modal statements have objective and nonvacuous truth-values that hold or fail independent of the mind, language, conventions, and such of the mathematical community. Mathematics comes out objective, even if it has no ontology. Chihara's program shows initial promise on the epistemic front. Perhaps it is easier to account for how the mathematician comes to know about what is possible, or about what sentences can be constructed, than it is to account for how the mathematician knows about a Platonic realm of objects. (See chapters 15 and 16 in this volume.)

4.4. Intuitionism

Unlike fictionalists, traditional *intuitionists*, such as L. E. J. Brouwer (e.g., [1912, 1948]) and Arend Heyting (e.g., [1930, 1956]), held that mathematics has a subject matter: mathematical objects, such as numbers, do exist. However, Brouwer and Heyting insisted that these objects are mind-dependent. Natural numbers and real numbers are mental constructions or are the result of mental constructions. In mathematics, to exist is to be constructed. Thus Brouwer and Heyting are anti-realists in ontology, denying the *objective* existence of mathematical objects. Some of their writing seems to imply that each person constructs his own mathematical realm. Communication between mathematicians consists in exchanging notes about their individual constructive activities. This would make mathematics subjective. It is more common, however, for these intuitionists, especially Brouwer, to hold that mathematics concerns the *forms* of mental construction as such (see Posy [1984]). This follows a Kantian theme, reviving the thesis that mathematics is synthetic a priori.

This perspective has consequences concerning the proper practice of mathematics. Most notably, the intuitionist demurs from the law of excluded middle—$(A \vee \neg A)$—and other inferences based on it. According to Brouwer and Heyting, these methodological principles are symptomatic of faith in the transcendental existence of mathematical objects and/or the transcendental truth of mathematical statements. For the intuitionist, every mathematical assertion must correspond to a construction. For example, let P be a property of natural numbers. For an intuitionist, the content of the assertion that not every number has the property P—the formula $\neg\forall x Px$—is that it is refutable that one can find a construction showing that P holds of each number. The content of the assertion that there is a number for which P fails—$\exists x\, \neg Px$—is that one can construct a number x and (p. 20) show that P does not hold of x. The latter formula cannot be inferred from the former because, clearly, it is possible to show that a property cannot hold universally without constructing a number for which it fails. In contrast, from the realist's perspective, the content of $\neg\forall x Px$ is simply that it is false that P holds universally, and $\exists x \neg Px$ means that there is a number for which P fails. Both formulas refer to numbers themselves; neither has anything to do with the knowledge-gathering abilities of mathematicians, or any other mental feature of them. From the realist's point of view, the two formulas are equivalent. The inference from $\neg\forall x Px$ to $\exists x \neg Px$ is a direct consequence of excluded middle.

Some contemporary intuitionists, such as Michael Dummett ([1973, 1977]) and Neil Tennant ([1987, 1997]), take a different route to roughly the same revisionist conclusion. Their proposed logic is similar to that of Brouwer and Heyting, but their supporting arguments and philosophy are different. Dummett begins with reflections on language acquisition and use, and the role of language in communication. One who understands a sentence must grasp its meaning, and one who learns a sentence thereby learns its meaning. As Dummett puts it, "a model of meaning is a model of understanding." This at least suggests that the meaning of a statement is somehow determined by its *use*. Someone who understands the meaning of any sentence of a language must be able to manifest that understanding in behavior. Since language is an instrument of communication, an individual cannot communicate what he cannot be observed to communicate.

Dummett argues that there is a natural route from this "manifestation requirement" to what we call here anti-realism in truth-value, and a route from there to the rejection of classical logic—and thus a demand for major revisions in mathematics.

Most semantic theories are *compositional* in the sense that the semantic content of a compound statement is analyzed in terms of the semantic content of its parts. Tarskian semantics, for example, is compositional, because the satisfaction of a complex formula is defined in terms of the satisfaction of its subformulas. Dummett's proposal is that the lessons of the manifestation requirement be incorporated into a compositional semantics. Instead of providing satisfaction conditions of each formula, Dummett proposes that the proper semantics supplies *proof* or *computation* conditions. He thus adopts what has been called "Heyting semantics." Here are three clauses:

> A proof of a formula in the form $\Phi \vee \Psi$ is a proof of Φ or a proof of Ψ.
>
> A proof of a formula in the form $\Phi \rightarrow \Psi$ is a procedure that can be proved to transform any proof of Φ into a proof of Ψ.
>
> A proof of a formula in the form $\neg\Phi$ is a procedure that can be proved to transform any proof of Φ into a proof of absurdity; a proof of $\neg\Phi$ is a proof that there can be no proof of Φ.

(p. 21)

Heyting and Dummett argue that on a semantics like this, the law of excluded middle is not universally upheld. A proof of a sentence of the form $\Phi \vee \neg\Phi$ consists of a proof of Φ or a proof that there can be no proof of Φ. The intuitionist claims that one cannot maintain this disjunction, in advance, for every sentence Φ.

A large body of research in mathematical logic shows how intuitionistic mathematics differs from its classical counterpart. Many mathematicians hold that the intuitionistic restrictions would cripple their discipline (see, e.g., Paul Bernays [1935]). For some philosophers of mathematics, the revision of mathematics is too high a price to pay. If a philosophy entails that there is something wrong with what the mathematicians do, then the philosophy is rejected out of hand. According to them, intuitionism can be safely ignored. A less dogmatic approach would be to take Dummett's arguments as a challenge to answer the criticisms he brings. Dummett argues that classical logic, and mathematics as practiced, do not enjoy a certain kind of justification, a justification one might think a logic and mathematics ought to have. Perhaps a defender of classical mathematics, such as a Quinean holist or a Maddy-style naturalist, can concede this, but argue that logic and mathematics do not need this kind of justification. We leave the debate at this juncture. (See chapters 9 and 10 in this volume.)

4.5. Structuralism

According to another popular philosophy of mathematics, the subject matter of arithmetic, for example, is the *pattern* common to any infinite system of objects that has a distinguished initial object, and a successor relation or operation that satisfies the induction principle. The arabic numerals exemplify this *natural number structure*, as do sequences of characters on a finite alphabet in lexical order, an infinite sequence of distinct moments of time, and so on. A natural number, such as 6, is a place in the natural number structure, the seventh place (if the structure starts with zero). Similarly, real analysis is about the real number structure, set theory is about the set-theoretic hierarchy structure, topology is about topological structures, and so on.

According to the structuralist, the application of mathematics to science occurs, in part, by discovering or postulating that certain structures are exemplified in the material world. Mathematics is to material reality as pattern is to patterned. Since a structure is a one-over-many of sorts, a structure is like a traditional universal, or property.

There are several ontological views concerning structures, corresponding roughly to traditional views concerning universals. One is that the natural number structure, for example, exists independent of whether it has instances in the (p. 22) physical world—or any other world, for that matter. Let us call this *ante rem* structuralism, after the analogous view concerning universals (see Shapiro [1997] and Resnik [1997]; see also Parsons [1990]). Another view is that there is no more to the natural number structure than the systems of objects that exemplify this structure. Destroy the systems, and the structure goes with them. From this perspective, either structures do not exist at all—in which case we have a version of nominalism—or the existence of structures is tied to the existence of their "instances," the systems that exemplify the structures. Views like this are sometimes dubbed *eliminative structuralism* (see Benacerraf [1965]).

According to *ante rem* structuralism, statements of mathematics are understood at face value. An apparent singular term, such as "2," is a genuine singular term, denoting a place in the natural number structure. For the eliminative structuralist, by contrast, these apparent singular terms are actually bound variables. For example, "2 + 3 = 5" comes to something like "in any natural number system *S*, any object in the 2-place of *S* that is *S*-added to the object in the 3-place of *S* is the object in the 5-place of *S*." Eliminative structuralism is a structuralism without structures.

Taken at face value, eliminative structuralism requires a large ontology to keep mathematics from being vacuous. For example, if there are only finitely many objects in the universe, then the natural number structure is not exemplified, and thus universally quantified statements of arithmetic are all vacuously true. Real and complex analysis and Euclidean geometry require a continuum of objects, and set theory requires a proper class (or at least an inaccessible cardinal number) of objects. For the *ante rem* structuralist, the structures themselves, and the places in the structures, provide the "ontology."

Benacerraf [1965], an early advocate of eliminative structuralism, made much of the fact that the set-theoretic hierarchy contains many exemplifications of the natural number structure. He concluded from this that numbers are not objects. This conclusion, however, depends on what it is to be an object—an interesting philosophical question in its own right. The *ante rem* structuralist readily accommodates the multiple realizability of the natural number structure: some items in the set-theoretic hierarchy, construed as objects, are organized into systems, and some of these systems exemplify the natural number structure. That is, *ante rem* structuralism accounts for the fact that mathematical structures are exemplified by other mathematical objects. Indeed, the natural number structure is exemplified by various *systems of natural numbers*, such as the even numbers and the prime numbers. From the *ante rem* perspective, this is straightforward: the natural numbers, as places in the natural number structure, exist. Some of them are organized into systems, and some of these systems exemplify the natural number structure.

On the *ante rem* view, the main epistemological question becomes: How do we know about structures? On the eliminative versions, the question is: How do (p. 23) we know about what holds in all systems of a certain type? Structuralists have developed several strategies for resolving the epistemic problems. The psychological mechanism of pattern recognition may be invoked for at least small, finite structures. By encountering instances of a given pattern, we obtain knowledge of the pattern itself. More sophisticated structures are apprehended via a Quine-style postulation (Resnik), and more robust forms of abstraction and implicit definition (Shapiro).

None of the structuralisms invoked so far provide for a reduction of the ontological burden of mathematics. The ontology of *ante rem* structuralism is as large and extensive as that of traditional realism in ontology. Indeed, *ante rem* structuralism *is* a realism in ontology. Only the nature of the ontology is in question. Eliminative structuralism also requires a large ontology to keep the various branches of mathematics from lapsing into vacuity. Surely there are not enough physical objects to keep structuralism from being vacuous when it comes to functional analysis or set theory. Thus, eliminative structuralism requires a large ontology of nonconcrete objects, and so it is not consistent with ontological anti-realism.

Hellman's [1989] *modal structuralism* is a variation of the underlying theme of eliminative structuralism which opts for a thorough ontological anti-realism. Instead of asserting that arithmetic is about all systems of a certain type, the modal structuralist says that arithmetic is about all *possible* systems of that type. A sum like "$2 + 3 = 5$" comes to "in any *possible* natural number system S, any object in the 2-place of S that is S-added to the object in the 3-place of S is the object in the 5-place of S" or "*necessarily*, in any natural number system S, any object in the 2-place of S that is S-added to the object in the 3-place of S is the object in the 5-place of S." The modal structuralist agrees with the eliminative structuralist that apparent singular terms, such as numerals, are disguised bound variables, but for the modal structuralist these variables occur inside the scope of a modal operator.

The modal structuralist faces an attenuated threat of vacuity similar to that of the eliminative structuralist. Instead of asserting that there are systems satisfying the natural number structure, for example, the modalist needs to affirm that such systems are possible. The key issue here is to articulate the underlying modality. (See chapters 17 and 18 in this volume.)

5. Logic

The above survey broached a number of issues concerning logic and the philosophy of logic. The debate over intuitionism invokes the general validity, within mathematics, of the law of excluded middle and other inferences based on it (p. 24) (chapters 9– 11 in this volume), and questions concerning impredicativity emerged from a version of logicism.

There is traffic in the other direction as well, from logic to the philosophy of mathematics. Perhaps the primary issue in the philosophy of logic concerns the nature, or natures, of logical consequence. There is, first, a deductive notion of consequence: a proposition Φ follows from a set Γ of propositions if Φ can be justified fully on the basis of the members of Γ. This is often understood in terms of a chain of legitimate, gap-free inferences that leads from members of Γ to Φ. A similar, perhaps identical, idea underlies Frege's development of logic in defense of logicism, and occurs also in neologicism. To show that a given mathematical proposition is knowable a priori and independent of intuition, we have to give a gap-free proof of it. There is also a semantic, model-theoretic notion of consequence: Φ follows from Γ if Φ is true in every interpretation (or model) of the language in which the members of Γ are true. Deductive systems introduced in logic books capture, or model, the deductive notion of consequence, and model-theoretic semantics captures, or tries to capture, the semantic notion.

There are substantial philosophical issues concerning the legitimacy of the model-theoretic notion of consequence and over which, if either, of the notions is primary. Of course, the resolution of these issues depends on prior questions concerning the nature of logic and the goals of logical study (chapters 21 and 22 in this volume). If both notions of consequence are legitimate, we can ask about relations between them. Surely it must be the case that if a proposition Φ follows deductively from a set Γ, then Φ is true under every interpretation of the language in which Γ is true. If not, then there is a chain of legitimate, gap-free inferences that can take us from truth to falsehood. Perish the thought. However, the converse seems less crucial. It may well be that there is a semantically valid argument whose conclusion cannot be deduced from its premises.

Issues surrounding higher-order logic, which were also broached briefly in the foregoing survey, turn on matters relating to the nature(s) of logical consequence. Second-order logic is inherently incomplete, in the sense that there is no effective deductive system that is both sound and complete for it. Does this disqualify it as logic, or is there some role for second-order logic to play? What does this say about the underlying nature of mathematics? (See chapters 25 and 26 in this volume).

Finally, there is a tradition, dating back to antiquity and very much alive today, that maintains that a proposition Φ cannot be a logical consequence of a set Γ unless Φ is somehow *relevant* to Γ. On the contemporary scene, the main targets of relevance logic are the so-called paradoxes of implication. One of these is the thesis that a logical truth follows from any set of premises whatsoever, and another is *ex falso quodlibet*, the thesis that any conclusion follows from a contradiction. The extent to which such inferences occur in mathematics is itself a subject of debate (chapters 23 and 24 in this volume.)

(p. 25)

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Stewart Shapiro

Stewart Shapiro is O'Donnell Professor of Philosophy at The Ohio State University and Professorial Fellow at the Arche Centre, University of St. Andrews.