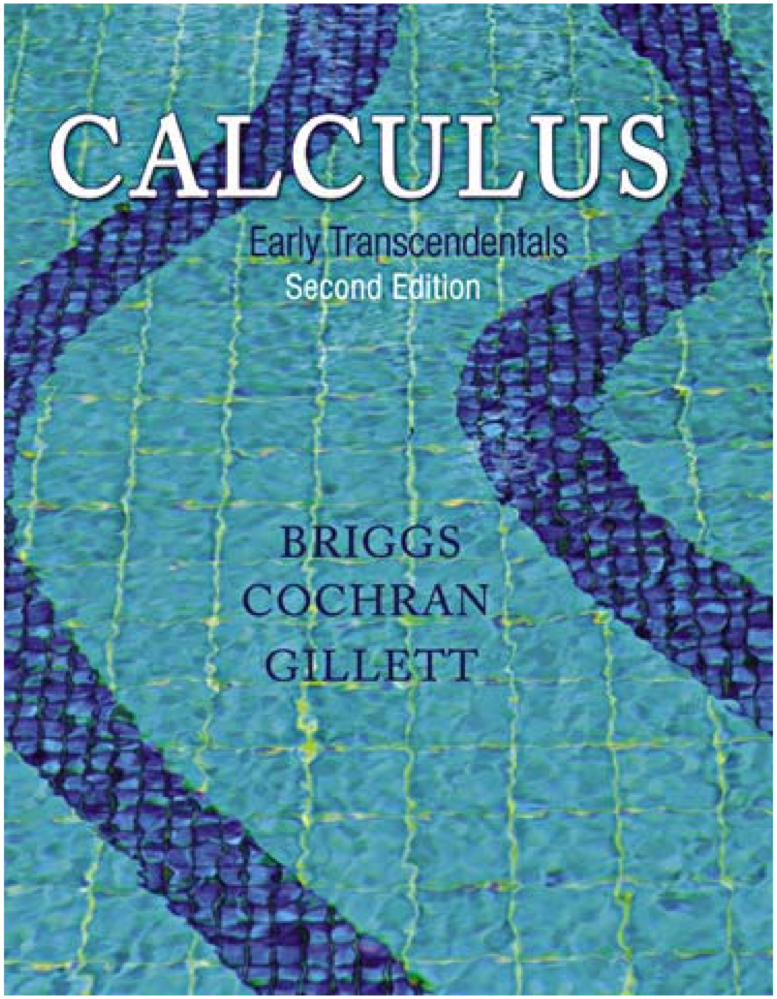
Chapter 4

Applications of
the Derivative



4.2

What Derivatives Tell Us

DEFINITION Increasing and Decreasing Functions

Suppose a function f is defined on an interval I. We say that f is **increasing** on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$. We say that f is **decreasing** on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.

Figure 4.13 (a & b)

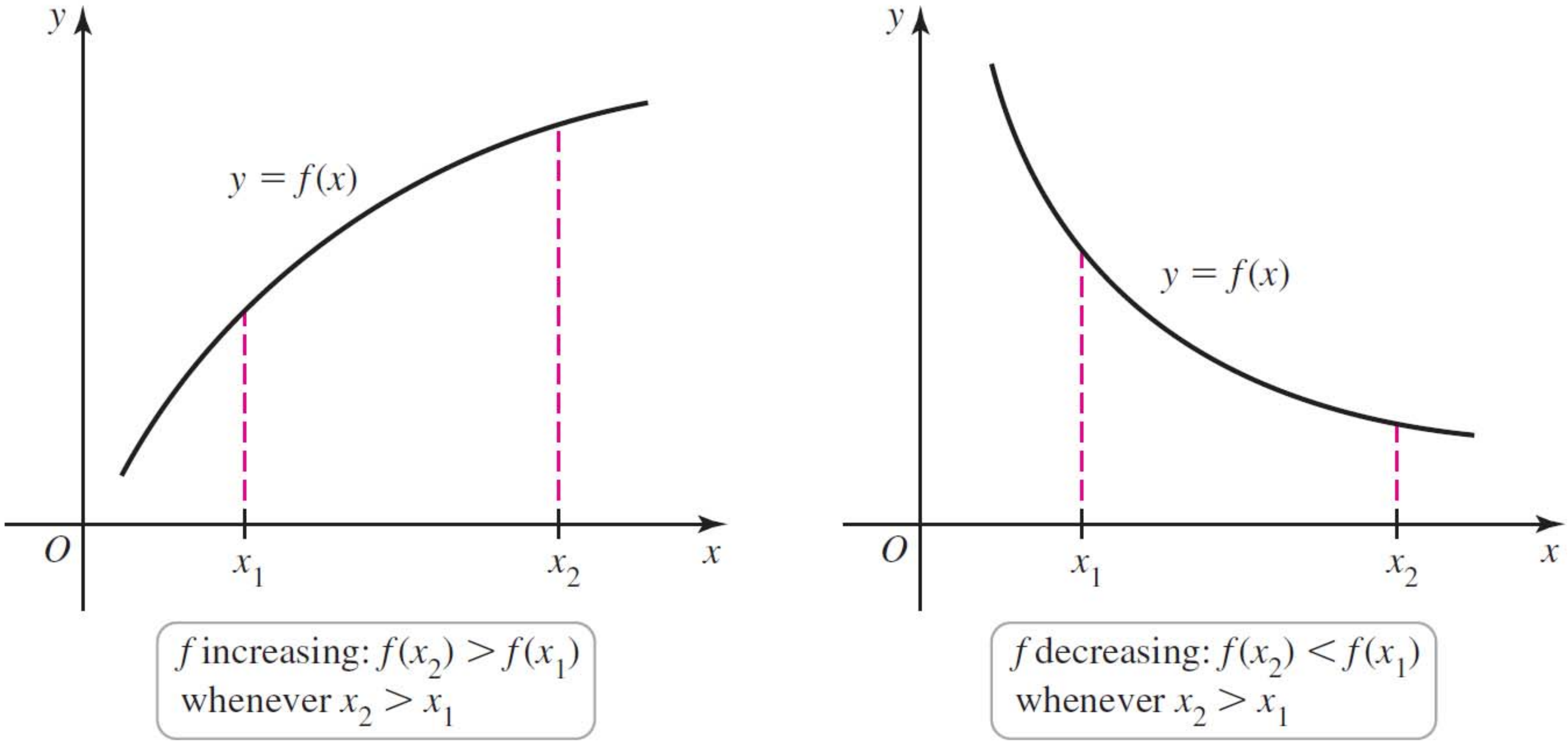
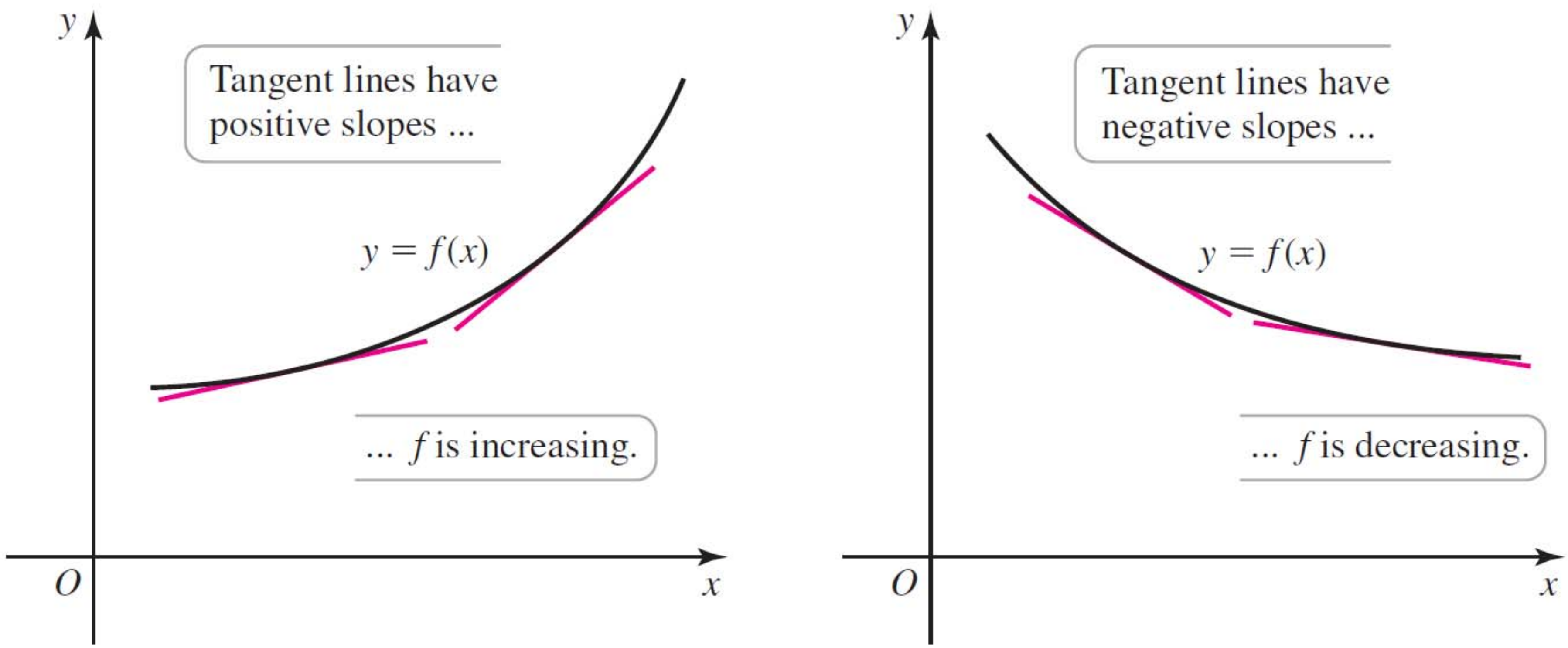


Figure 4.14



y
Tangent lines have positive slopes ...
y = f(x)
... f is increasing.
O
x
y
Tangent lines have negative slopes ...
y = f(x)
... f is decreasing.
O
x

THEOREM 4.3 Test for Intervals of Increase and Decrease

Suppose f is continuous on an interval I and differentiable at all interior points of I. If $f'(x) > 0$ at all interior points of I, then f is increasing on I. If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.

Figure 4.15

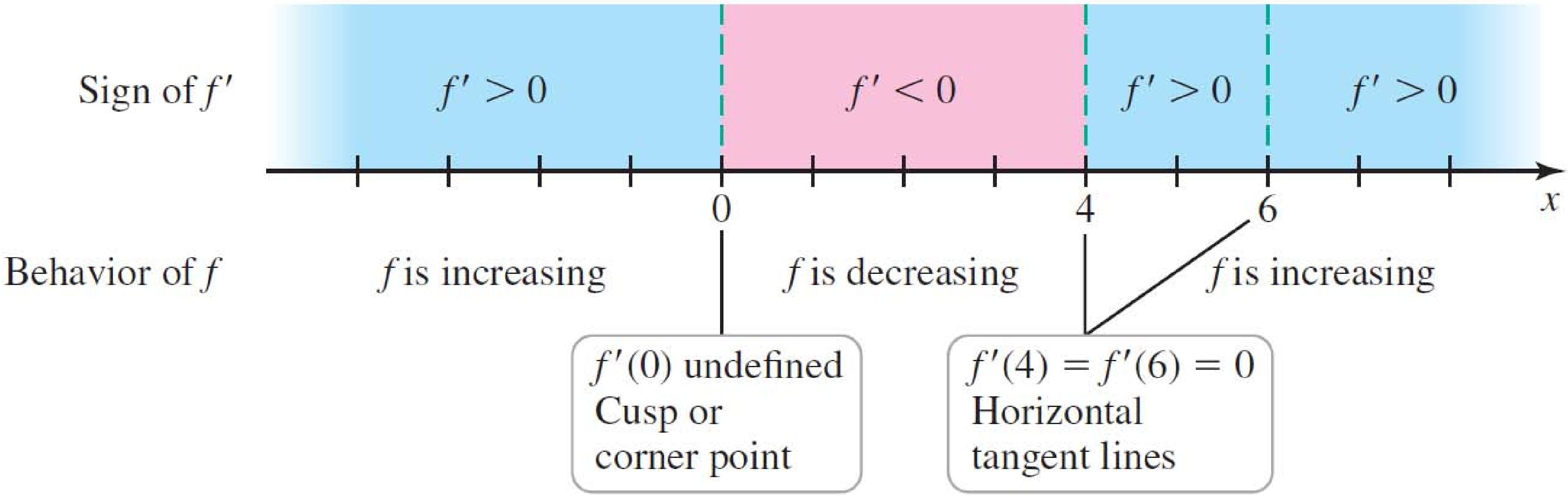

Sign of f'
$f' > 0$
$f' < 0$
$f' > 0$
$f' > 0$
0
4
6
x
Behavior of f
f is increasing
f is decreasing
f is increasing
$f'(0)$ undefined
Cusp or
corner point
$f'(4) = f'(6) = 0$
Horizontal
tangent lines

Figure 4.16

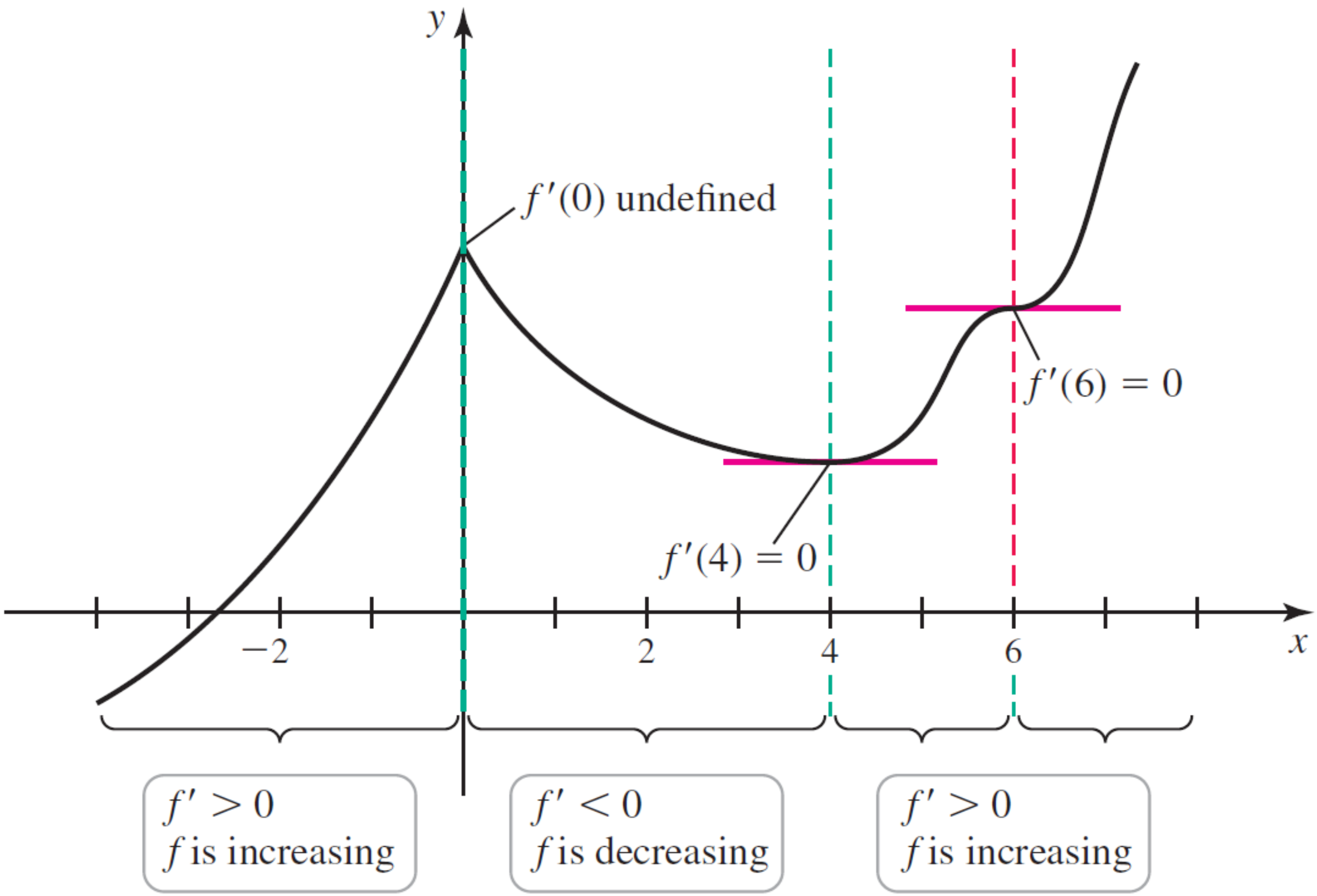

y
f′(0) undefined
f′(6) = 0
f′(4) = 0
−2
2
4
6
x
f′ > 0
f is increasing
f′ < 0
f is decreasing
f′ > 0
f is increasing

Figure 4.17

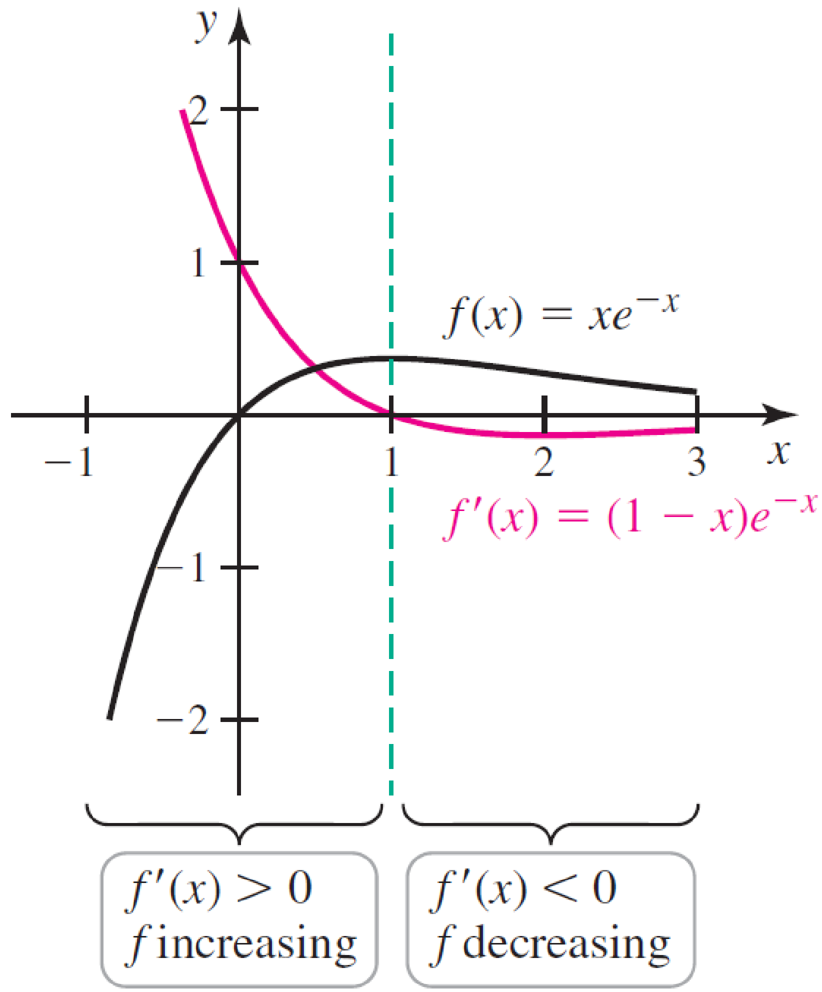
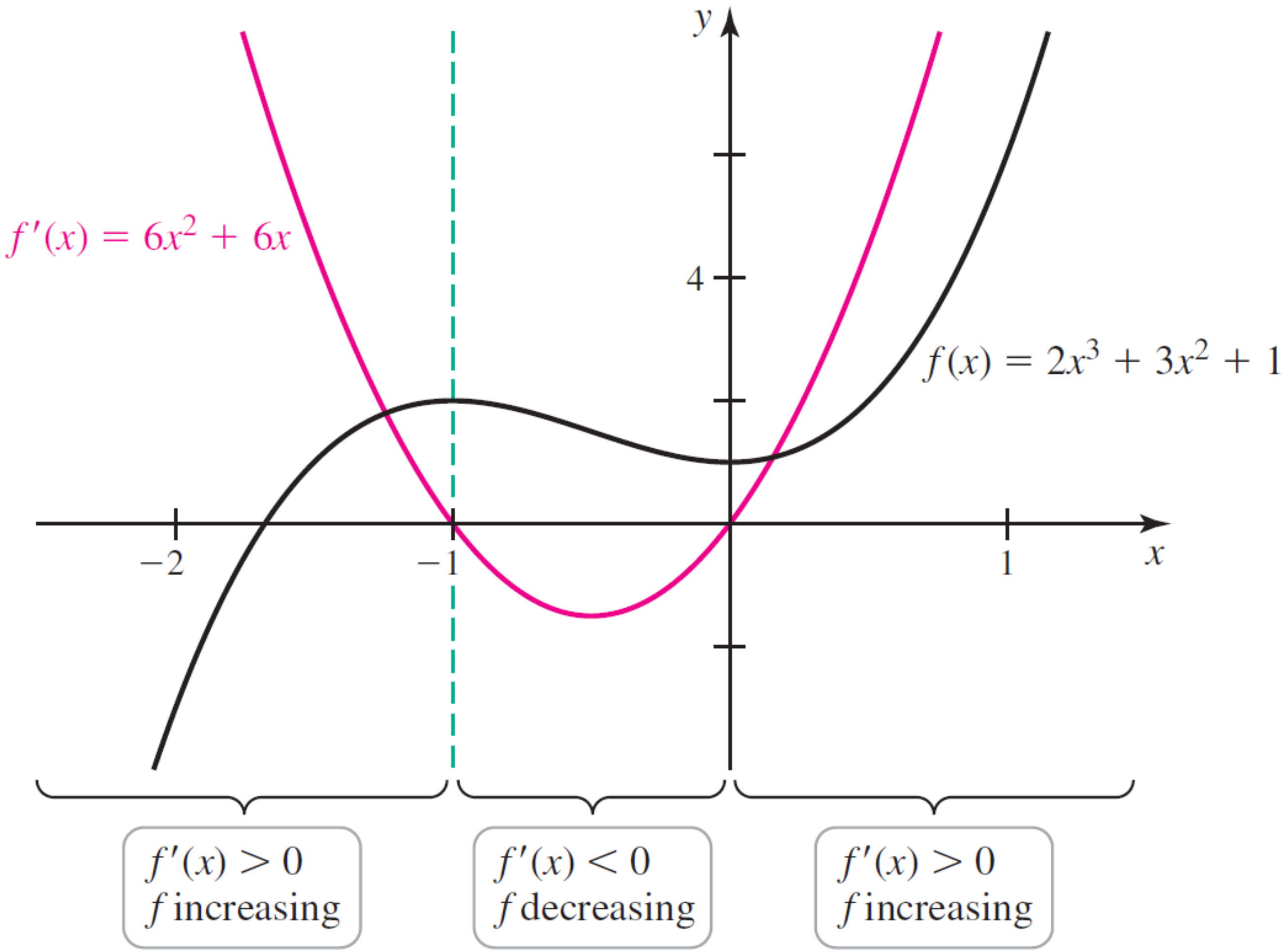


Figure 4.18



y
$f'(x) = 6x^2 + 6x$
4
$f(x) = 2x^3 + 3x^2 + 1$
−2
−1
1
x
$f'(x) > 0$
f increasing
$f'(x) < 0$
f decreasing
$f'(x) > 0$
f increasing

Figure 4.19 (a)

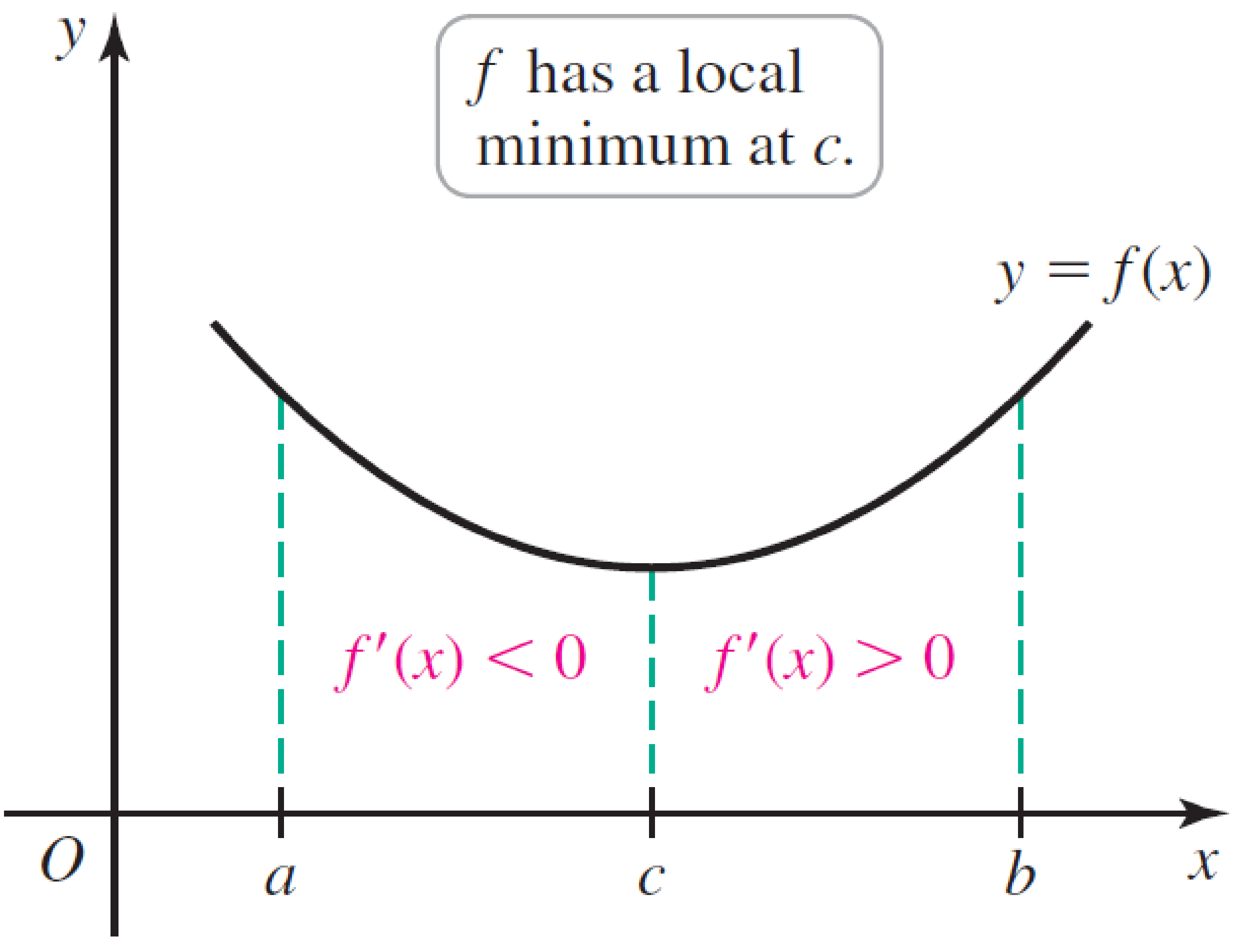


Figure 4.19 (b)

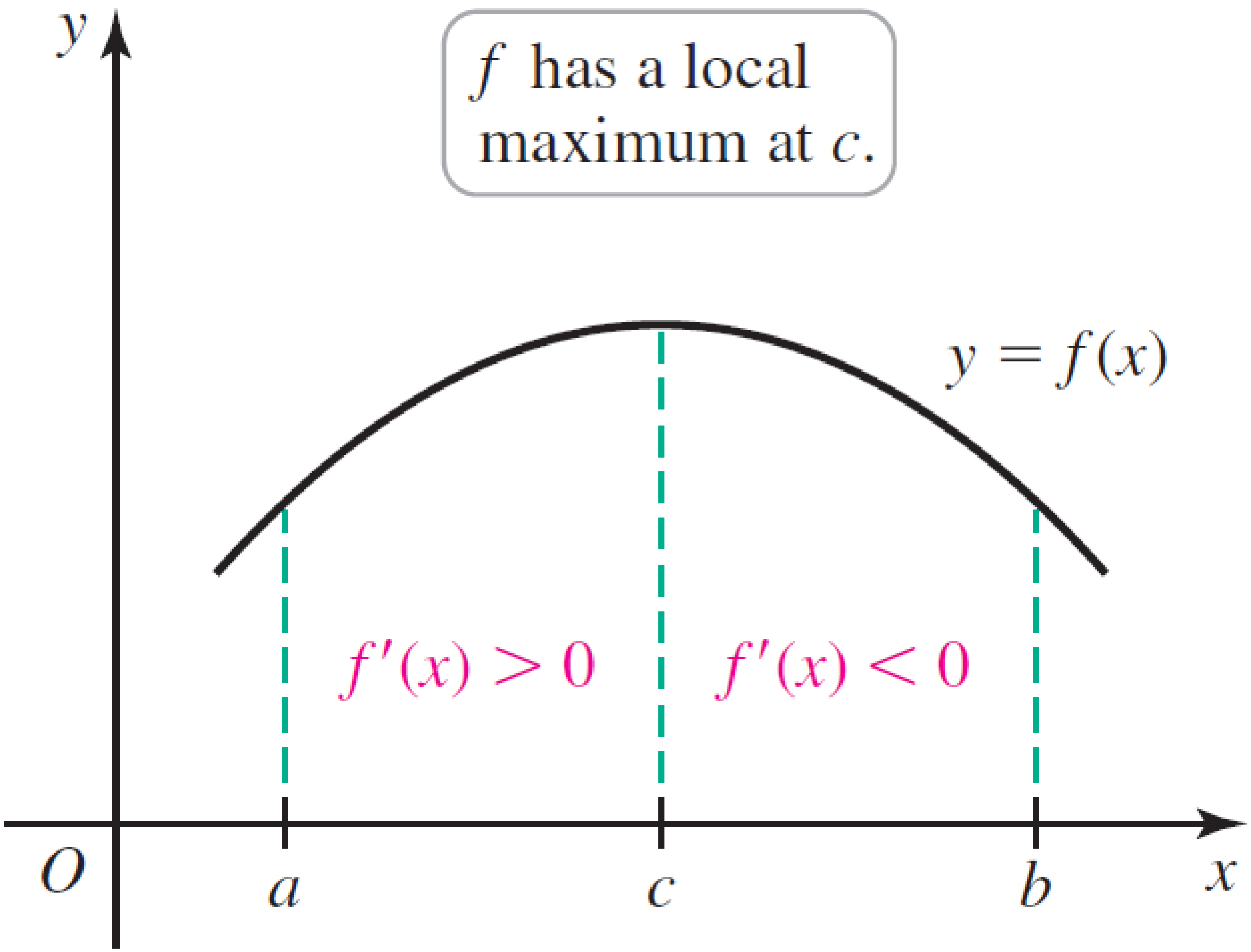
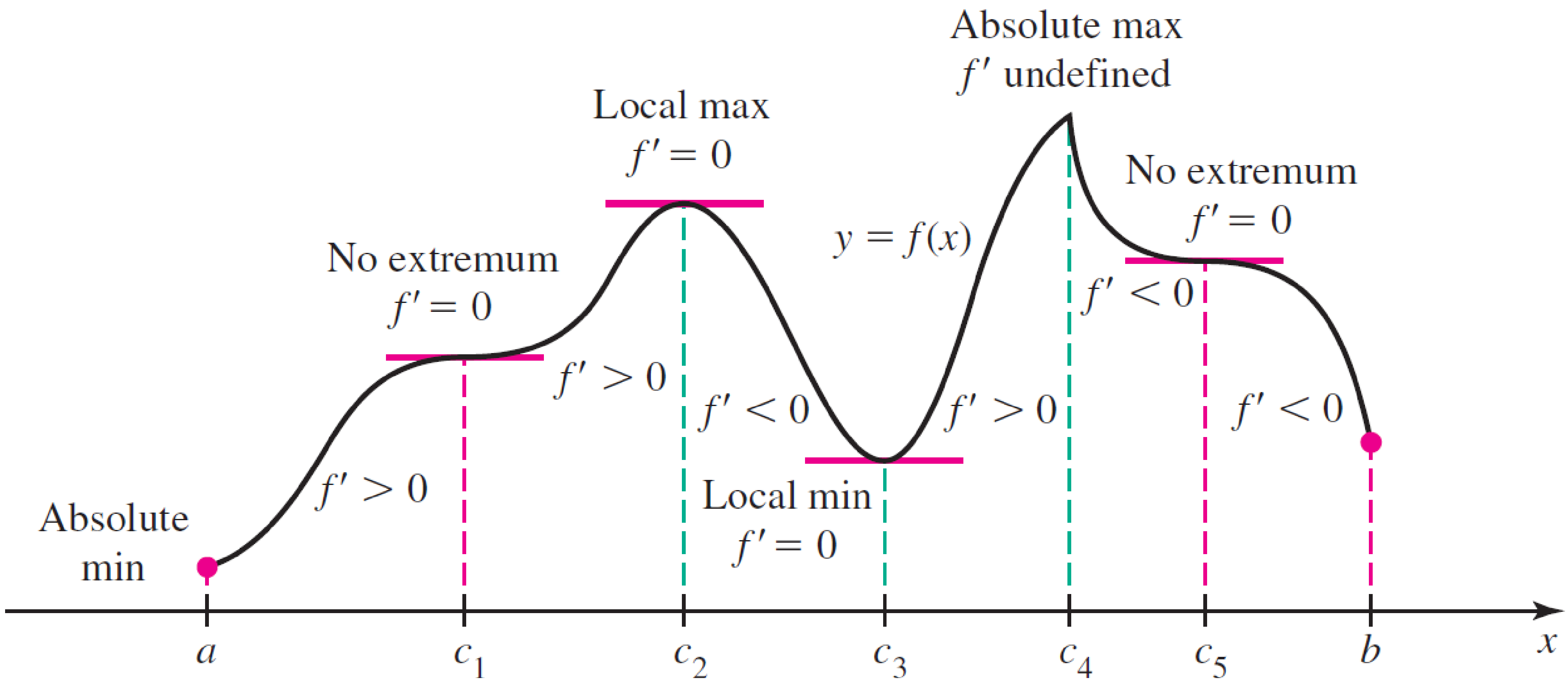


Figure 4.20


Absolute max
f′ undefined
Local max
f′ = 0
No extremum
f′ = 0
y = f(x)
No extremum
f′ = 0
f′ > 0
f′ < 0
f′ > 0
f′ < 0
f′ < 0
f′ > 0
Absolute
min
Local min
f′ = 0
a
c1
c2
c3
c4
c5
b
x

THEOREM 4.4 First Derivative Test

Suppose that f is continuous on an interval that contains a critical point c and assume f is differentiable on an interval containing c, except perhaps at c itself.

- If f' changes sign from positive to negative as x increases through c, then f has a **local maximum** at c.
- If f' changes sign from negative to positive as x increases through c, then f has a **local minimum** at c.
- If f' is positive on both sides near c or negative on both sides near c, then f has no local extreme value at c.

Figure 4.21

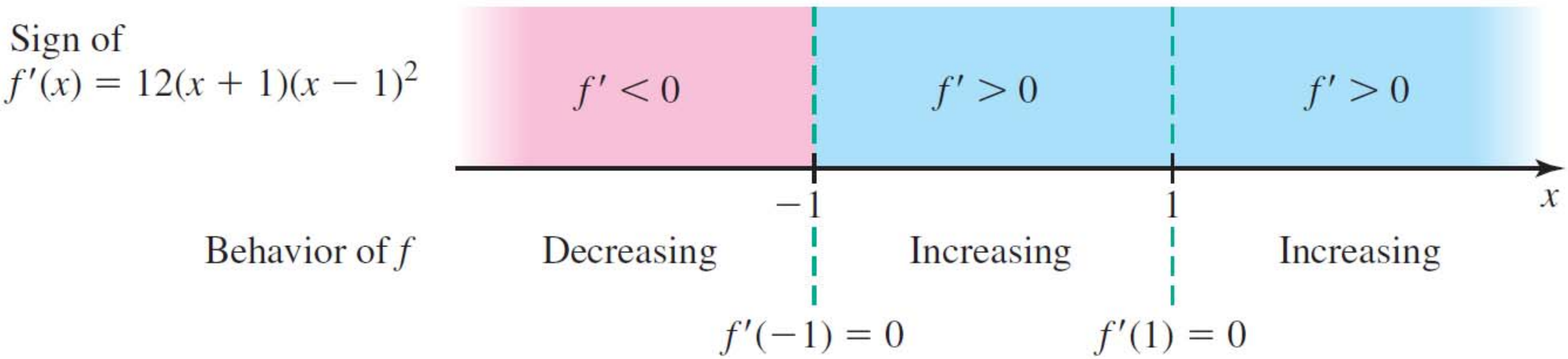
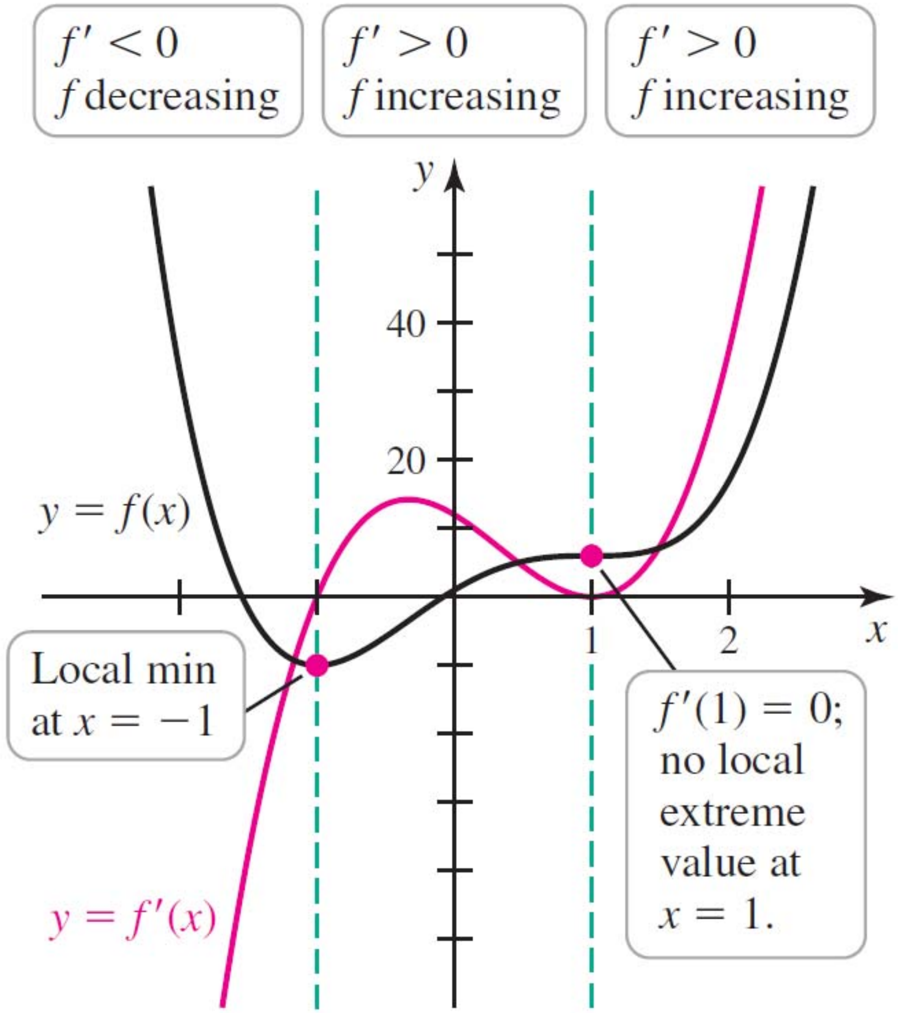

Sign of
$f'(x) = 12(x+1)(x-1)^2$
$f' < 0$
$f' > 0$
$f' > 0$
-1
1
x
Behavior of f
Decreasing
Increasing
Increasing
$f'(-1) = 0$
$f'(1) = 0$

Figure 4.22



$f' < 0$
f decreasing
$f' > 0$
f increasing
$f' > 0$
f increasing
y
40
20
$y = f(x)$
1
2
x
Local min
at $x = -1$
$f'(1) = 0$;
no local
extreme
value at
$x = 1$.
$y = f'(x)$

Figure 4.23

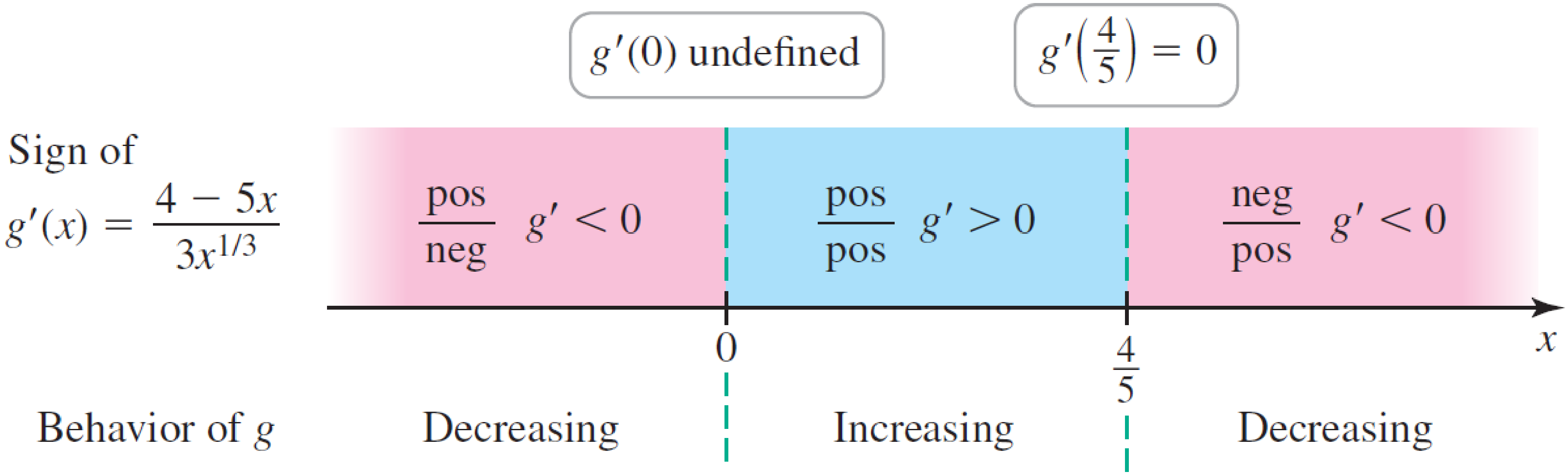
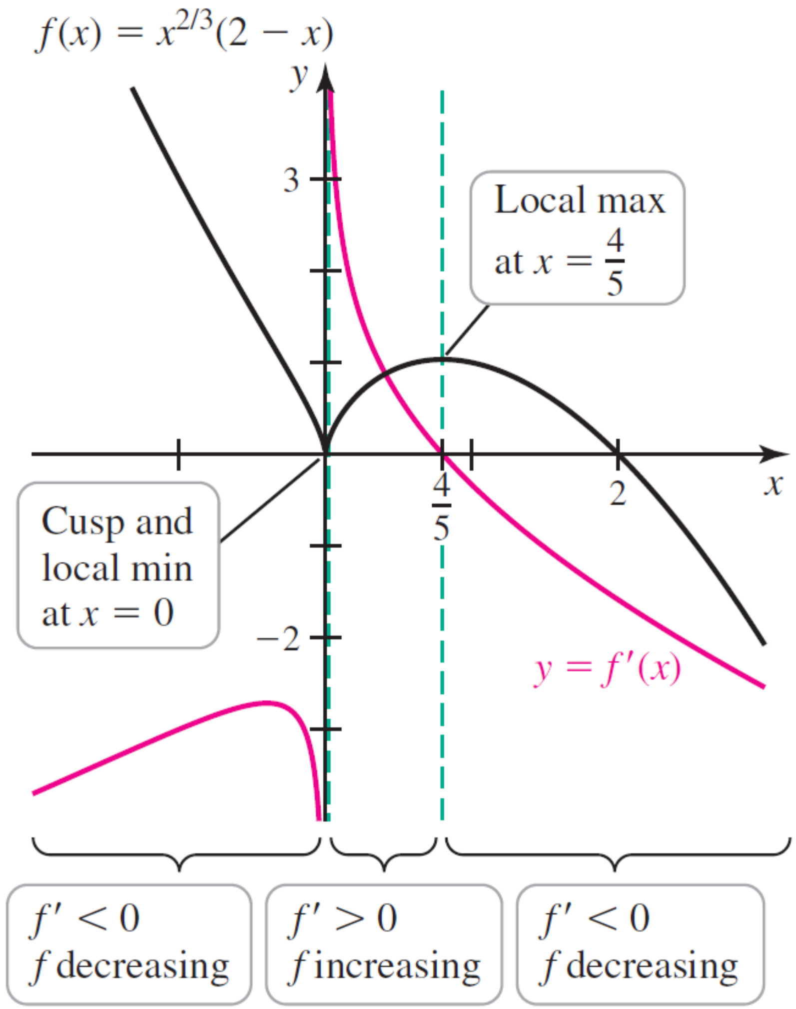

$g'(0)$ undefined
$g'\left(\frac{4}{5}\right) = 0$
Sign of
$g'(x) = \frac{4 - 5x}{3x^{1/3}}$
$\frac{\text{pos}}{\text{neg}}$ $g' < 0$
$\frac{\text{pos}}{\text{pos}}$ $g' > 0$
$\frac{\text{neg}}{\text{pos}}$ $g' < 0$
0
$\frac{4}{5}$
x
Behavior of g
Decreasing
Increasing
Decreasing

Figure 4.24


$f(x) = x^{2/3}(2 - x)$
y
3
−2
x
$\frac{4}{5}$
2
Local max at $x = \frac{4}{5}$
Cusp and local min at $x = 0$
$y = f'(x)$
$f' < 0$ f decreasing
$f' > 0$ f increasing
$f' < 0$ f decreasing

THEOREM 4.5 One Local Extremum Implies Absolute Extremum

Suppose f is continuous on an interval I that contains exactly one local extremum at c.

- If a local maximum occurs at c, then $f(c)$ is the absolute maximum of f on I.
- If a local minimum occurs at c, then $f(c)$ is the absolute minimum of f on I.

Figure 4.25

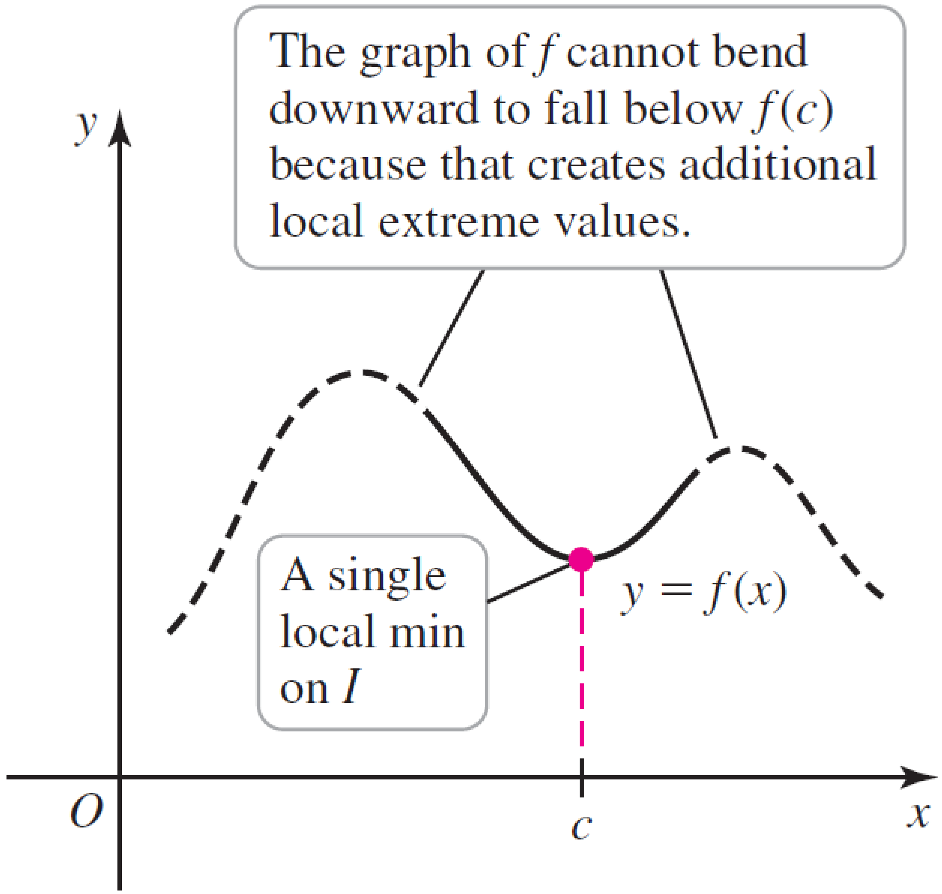

The graph of f cannot bend downward to fall below $f(c)$ because that creates additional local extreme values.
A single local min on I
$y = f(x)$
y
x
O
c

Figure 4.26

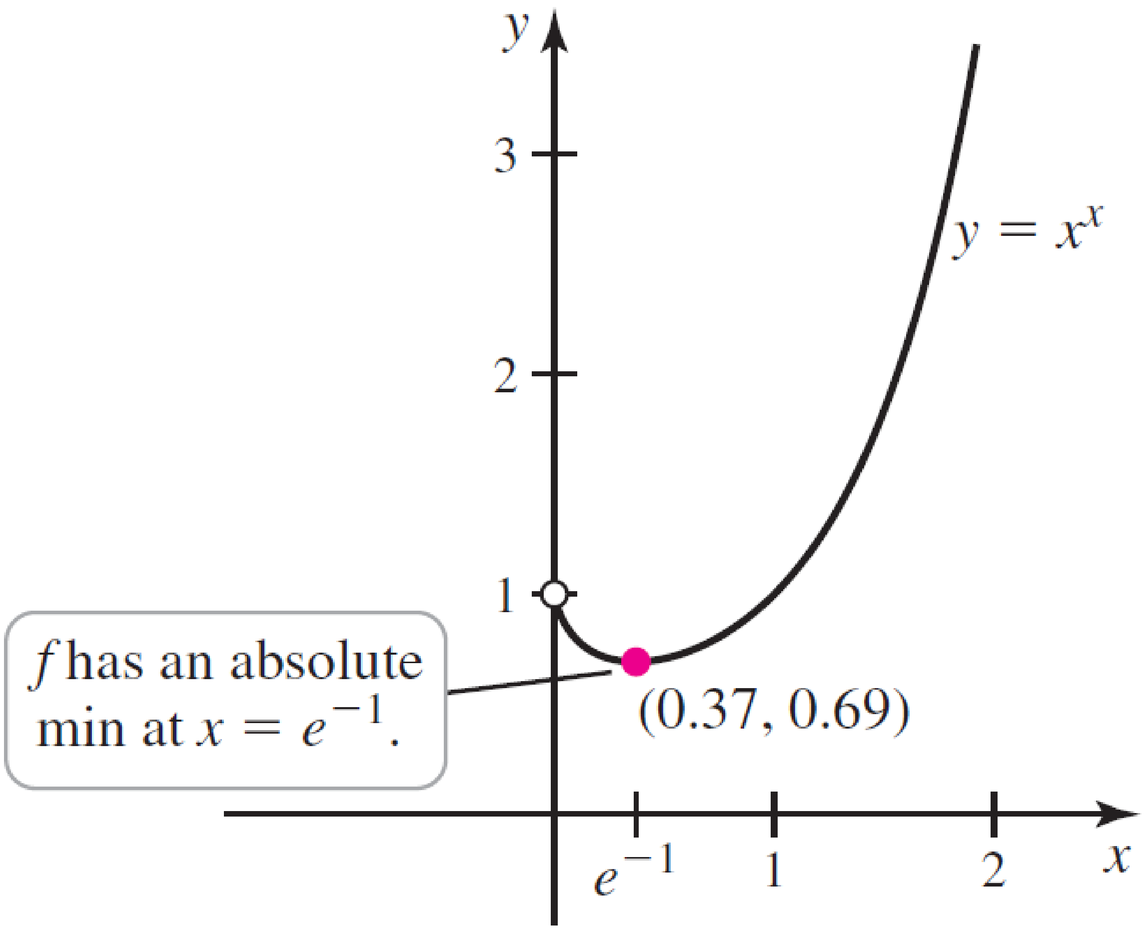
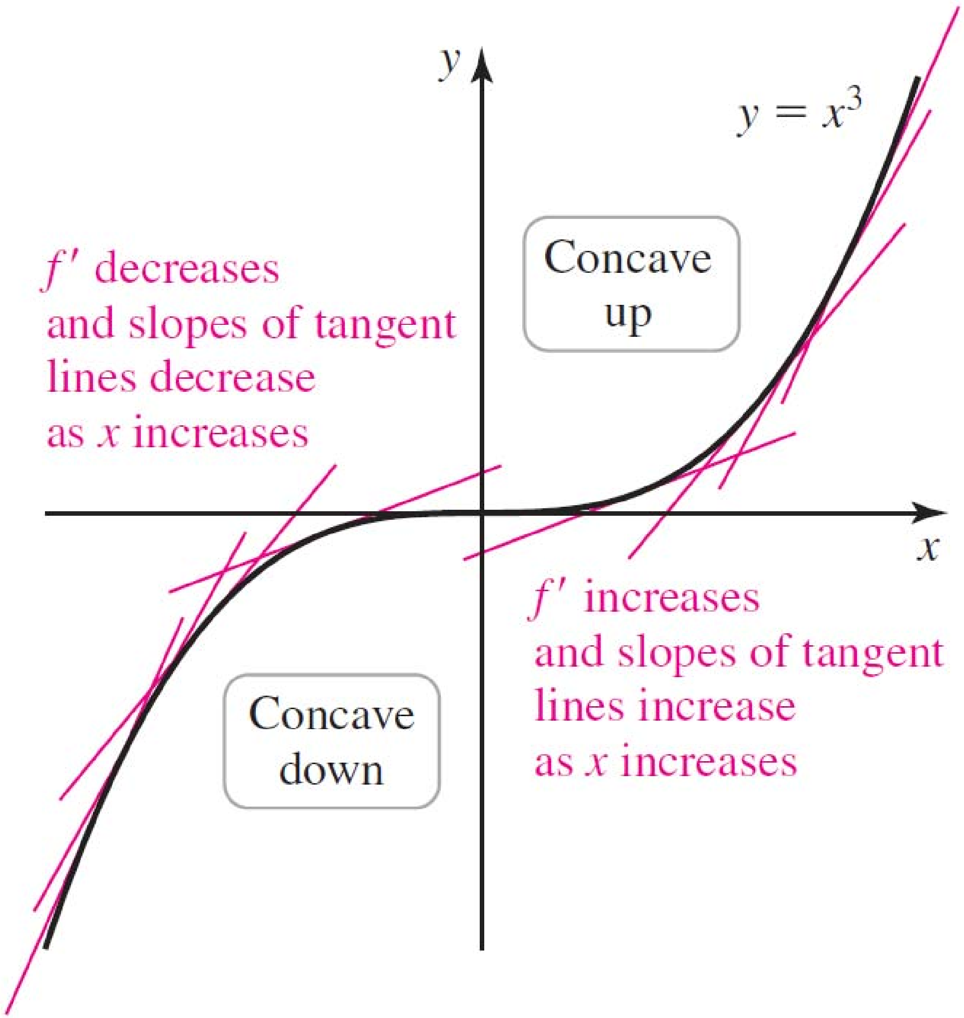


Figure 4.27


y
y = x³
x
Concave
up
Concave
down
f′ decreases
and slopes of tangent
lines decrease
as x increases
f′ increases
and slopes of tangent
lines increase
as x increases

DEFINITION Concavity and Inflection Point

Let f be differentiable on an open interval I. If f' is increasing on I, then f is **concave up** on I. If f' is decreasing on I, then f is **concave down** on I.

If f is continuous at c and f changes concavity at c (from up to down, or vice versa), then f has an **inflection point** at c.

THEOREM 4.6 Test for Concavity

Suppose that f'' exists on an open interval I.

- If $f'' > 0$ on I, then f is concave up on I.
- If $f'' < 0$ on I, then f is concave down on I.
- If c is a point of I at which f'' changes sign at c (from positive to negative, or vice versa), then f has an inflection point at c.

Figure 4.28 (a & b)

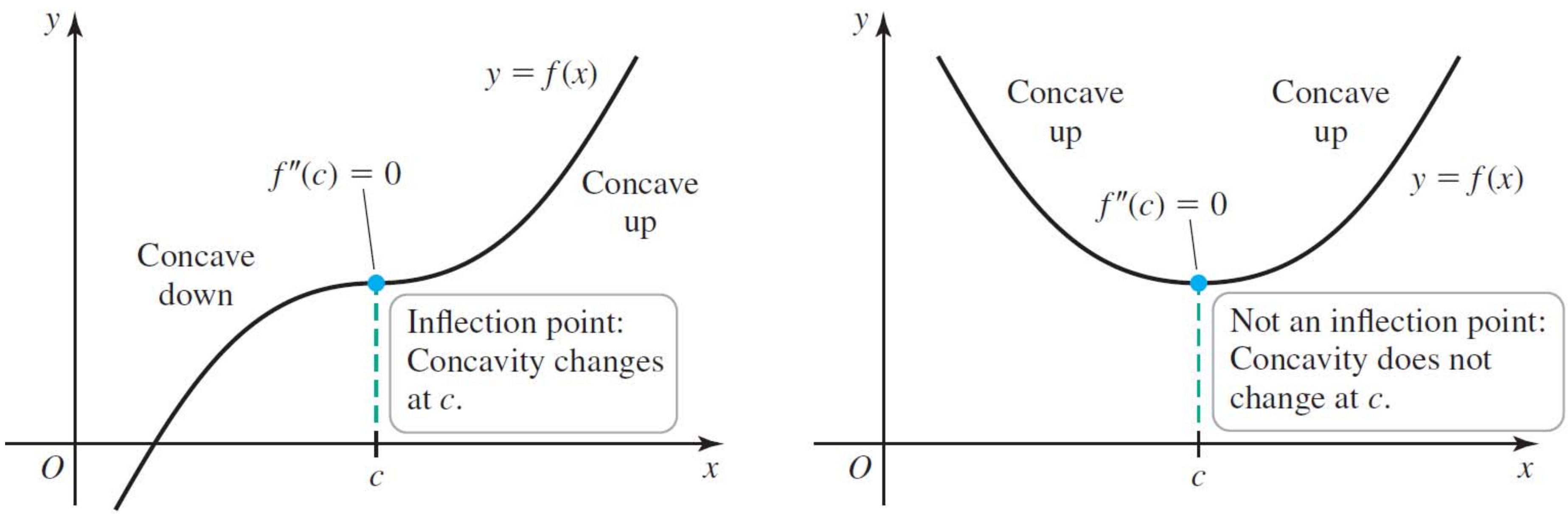
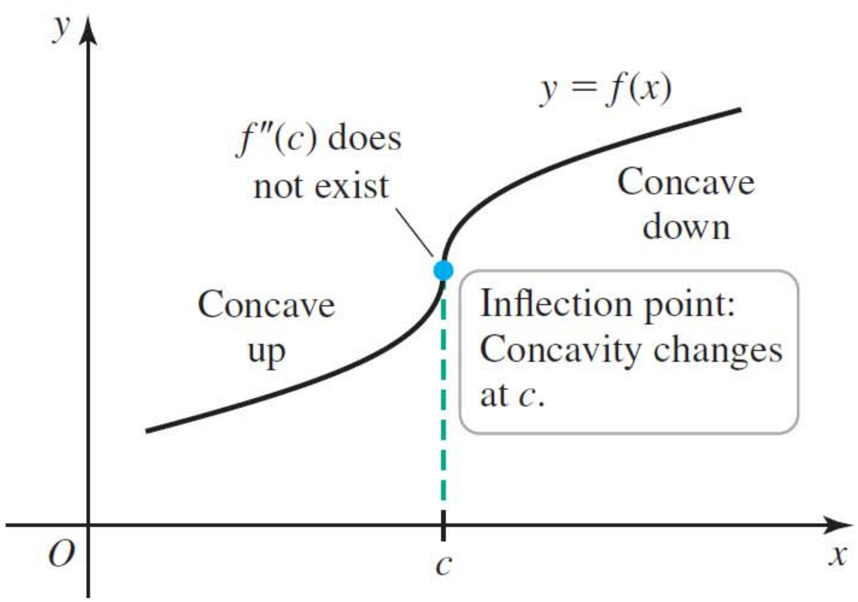


Figure 4.28 (c & d)



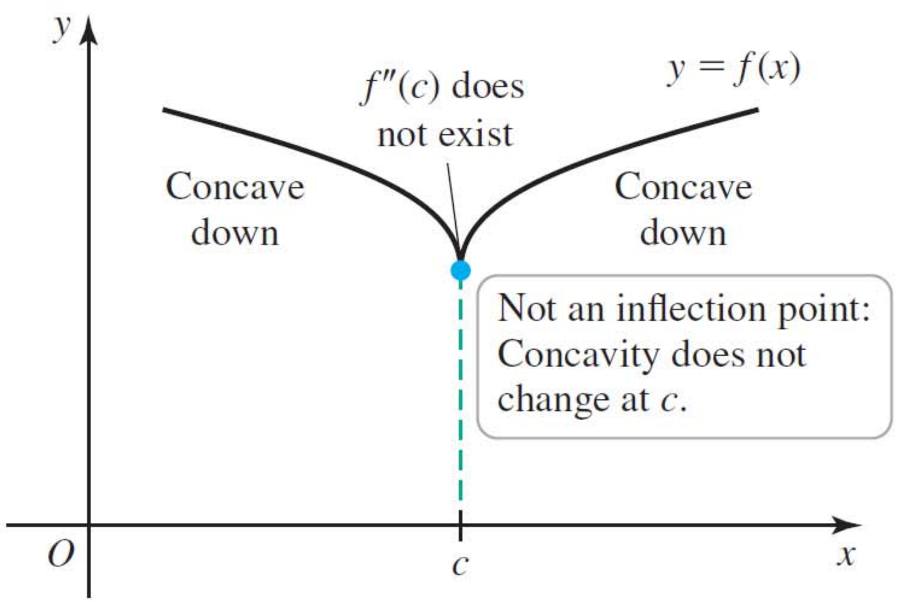


Figure 4.29 (a)

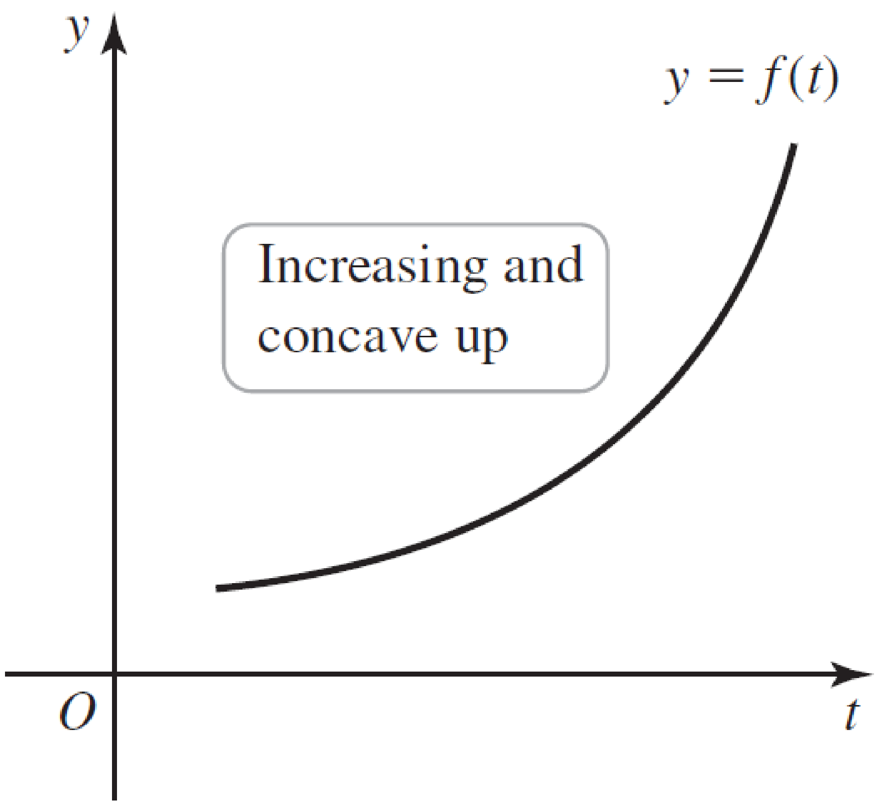


Figure 4.29 (b)

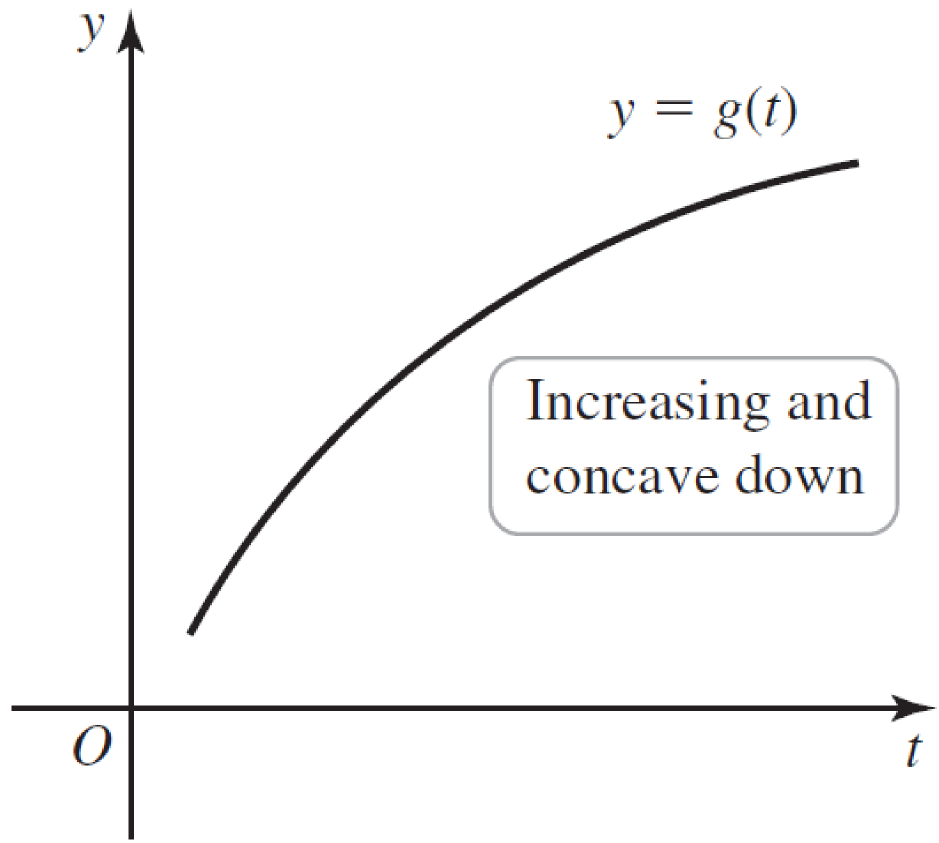


Figure 4.30

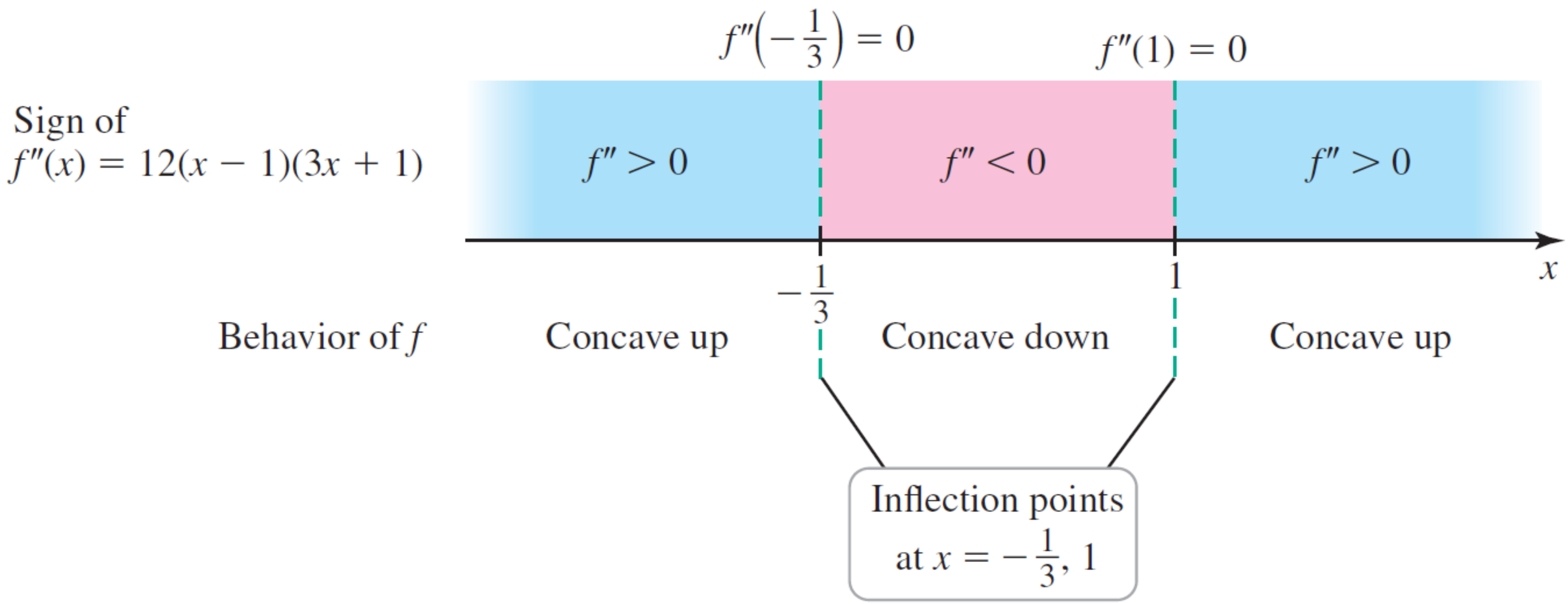
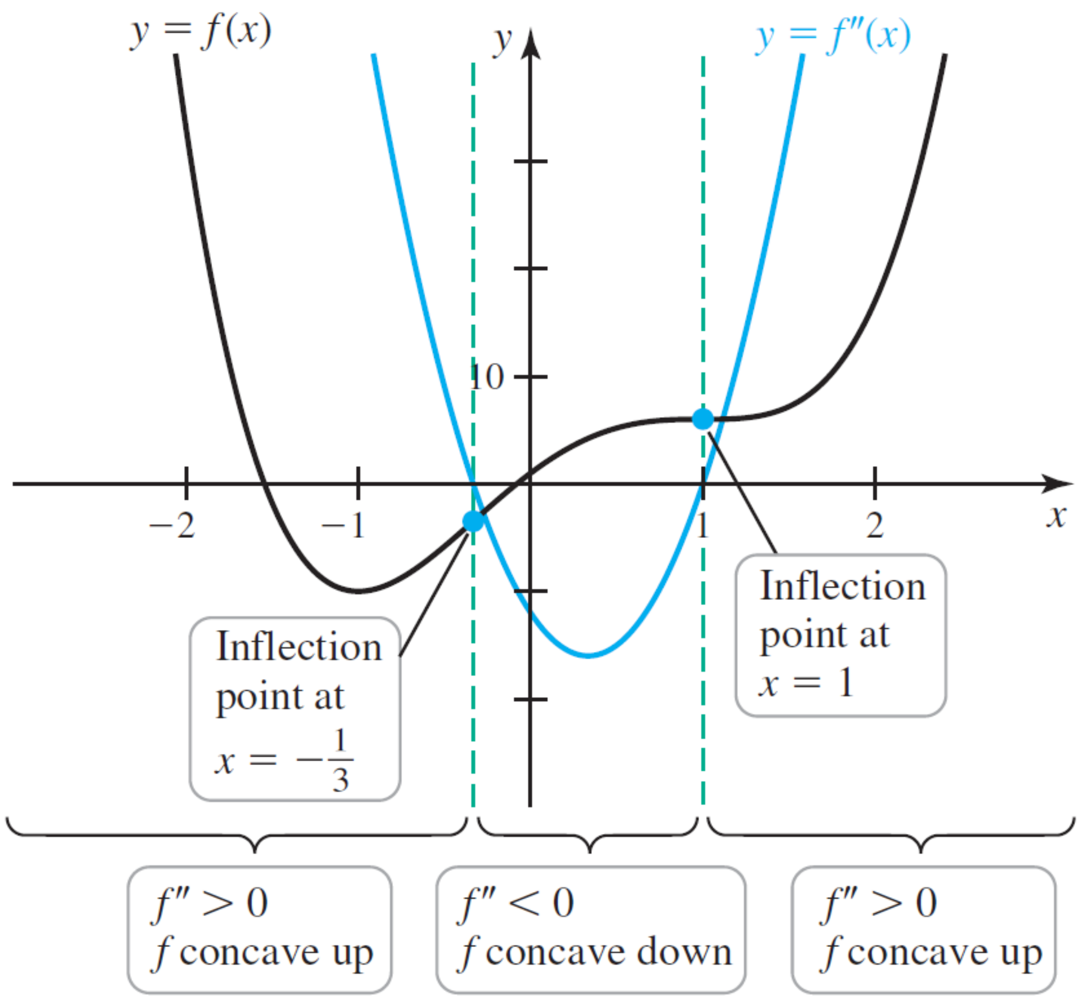
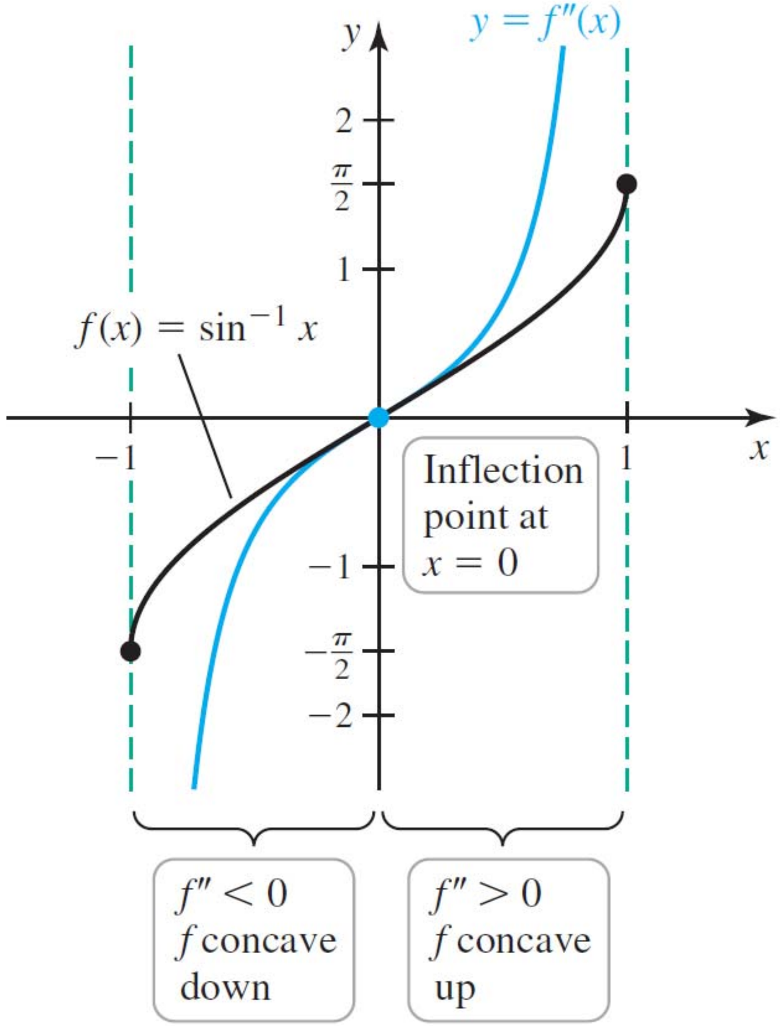

$f''\left(-\frac{1}{3}\right) = 0$
$f''(1) = 0$
Sign of
$f''(x) = 12(x - 1)(3x + 1)$
$f'' > 0$
$f'' < 0$
$f'' > 0$
$-\frac{1}{3}$
1
x
Behavior of f
Concave up
Concave down
Concave up
Inflection points
at $x = -\frac{1}{3}, 1$

Figure 4.31



$y = f(x)$
y
$y = f''(x)$
10
−2
−1
1
2
x
Inflection point at $x = -\frac{1}{3}$
Inflection point at $x = 1$
$f'' > 0$ f concave up
$f'' < 0$ f concave down
$f'' > 0$ f concave up

Figure 4.32



y
$y = f''(x)$
2
$\frac{\pi}{2}$
1
$f(x) = \sin^{-1} x$
-1
1
x
Inflection point at $x = 0$
-1
$-\frac{\pi}{2}$
-2
$f'' < 0$ f concave down
$f'' > 0$ f concave up

THEOREM 4.7 Second Derivative Test for Local Extrema

Suppose that f'' is continuous on an open interval containing c with $f'(c) = 0$.

- If $f''(c) > 0$, then f has a local minimum at c (Figure 4.33a).
- If $f''(c) < 0$, then f has a local maximum at c (Figure 4.33b).
- If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, local minimum, or neither at c.

Figure 4.33 (a)

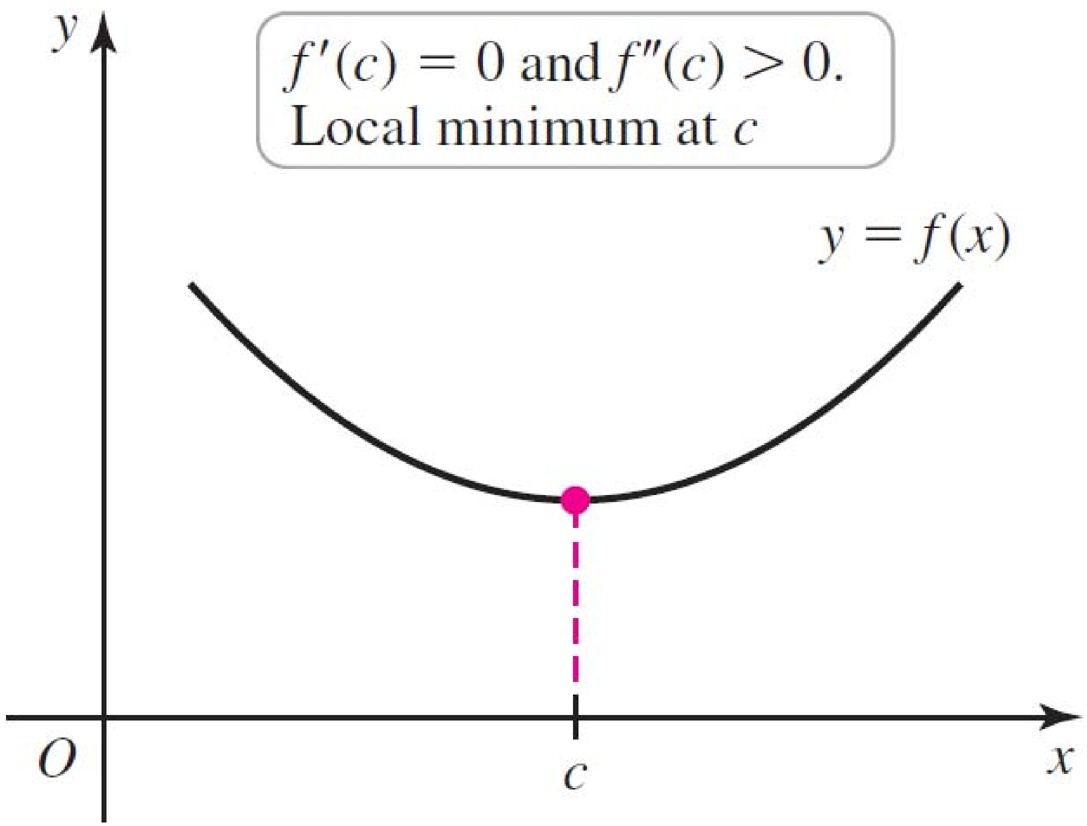


Figure 4.33 (b)

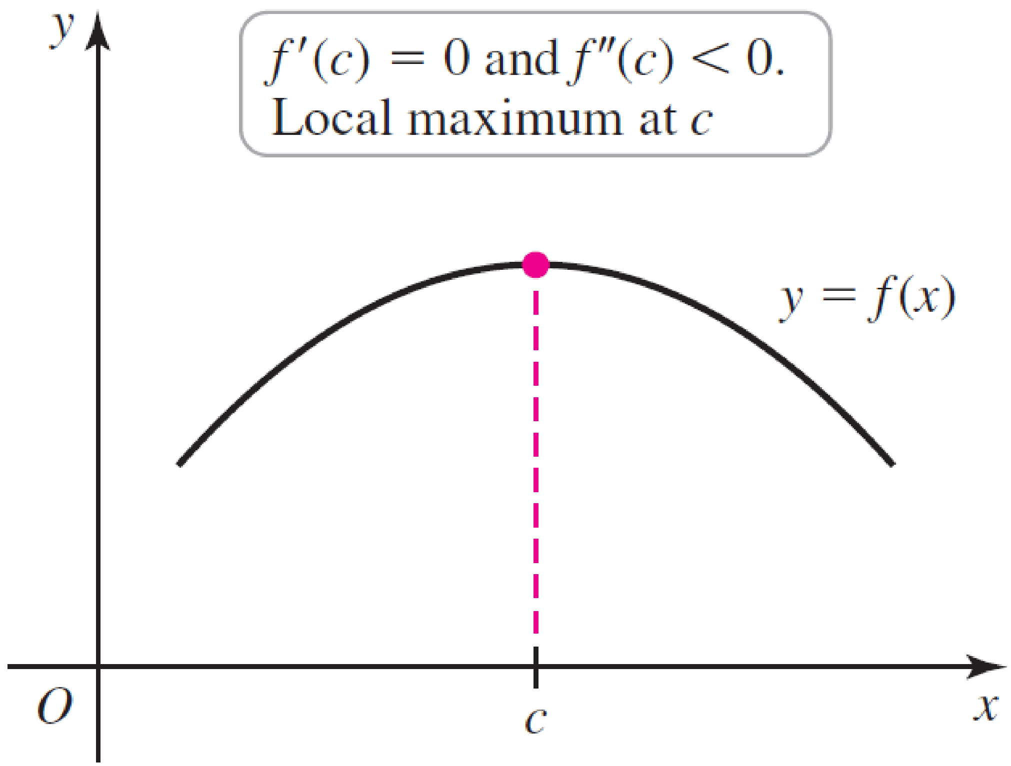
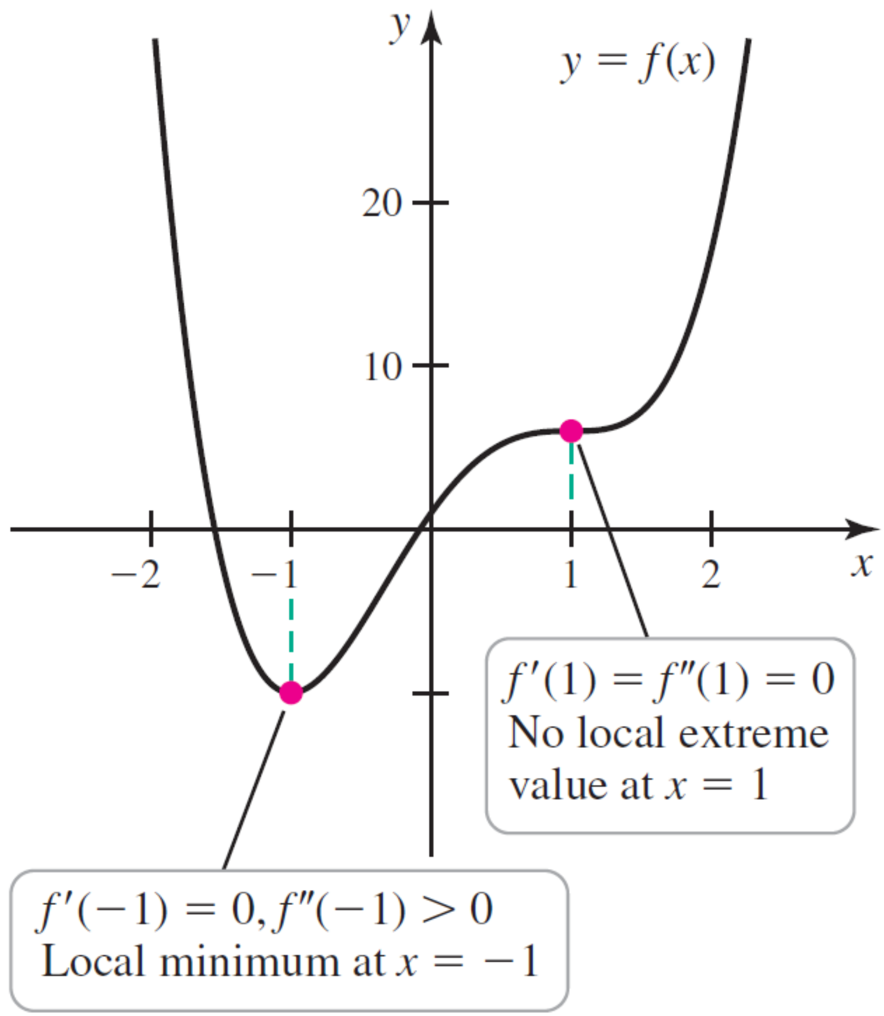
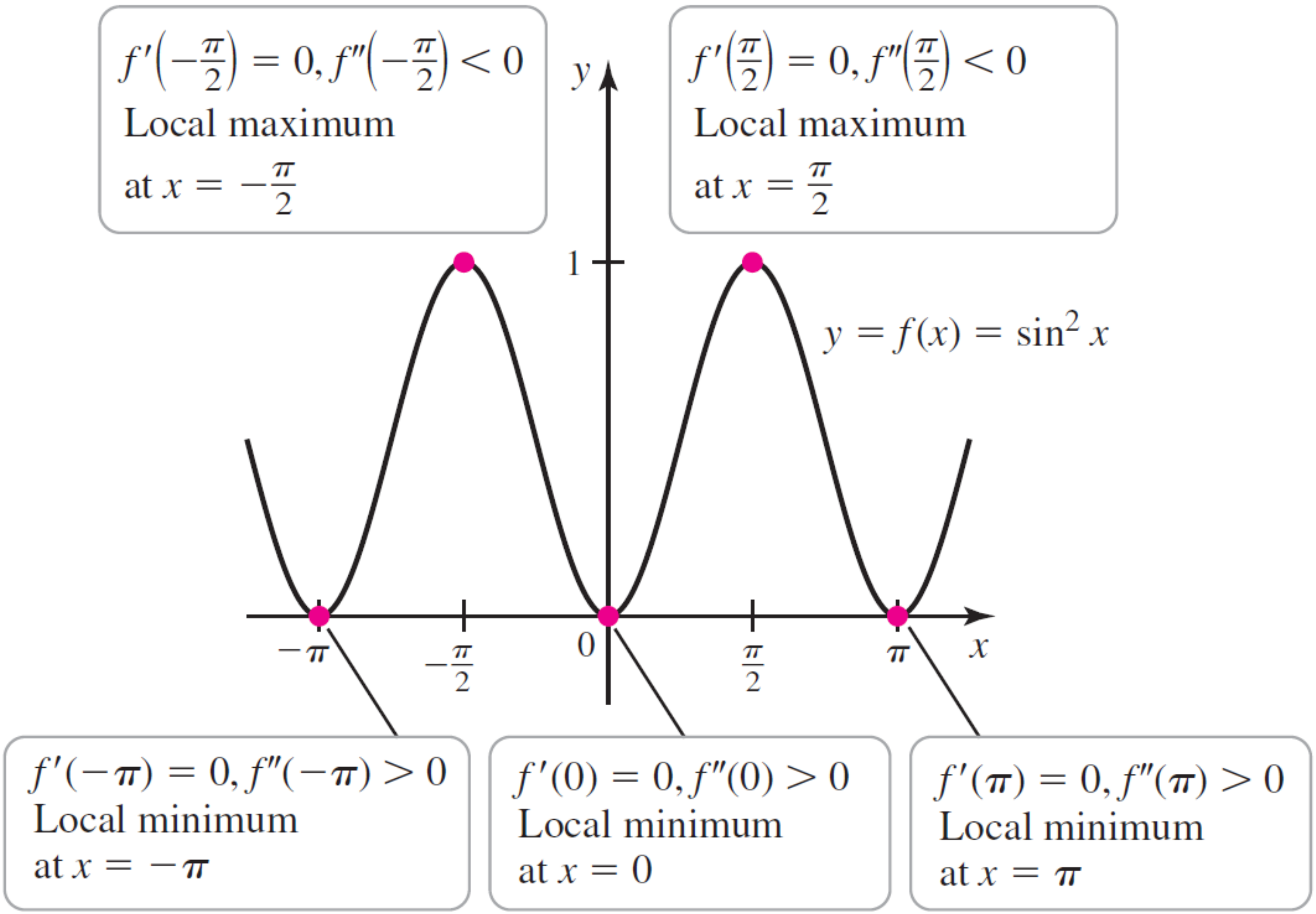


Figure 4.34



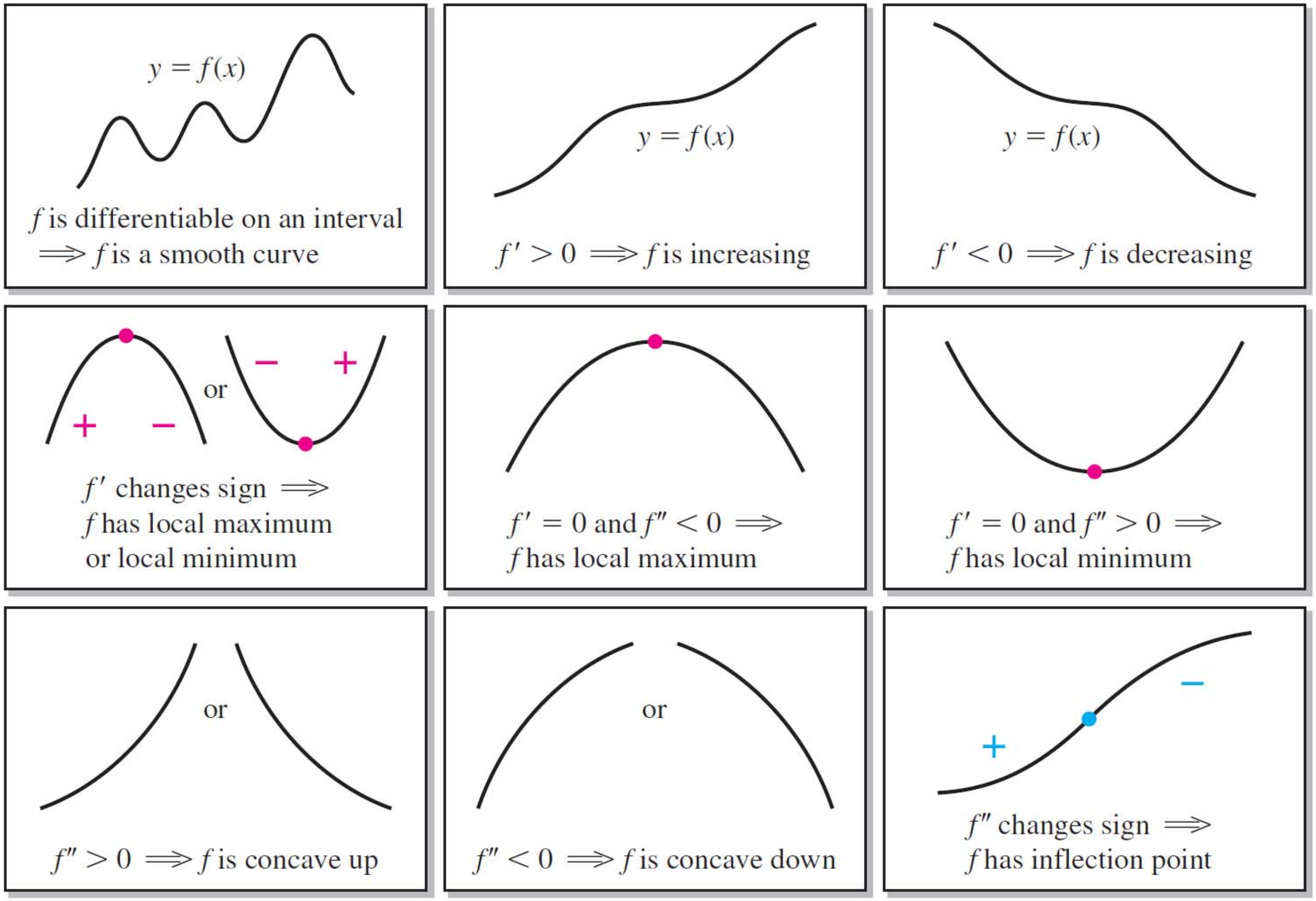
y
y = f(x)
20
10
−2
−1
1
2
x
f′(1) = f″(1) = 0
No local extreme
value at x = 1
f′(−1) = 0, f″(−1) > 0
Local minimum at x = −1

Figure 4.35



$f'\left(-\frac{\pi}{2}\right) = 0, f''\left(-\frac{\pi}{2}\right) < 0$
Local maximum
at $x = -\frac{\pi}{2}$
$f'\left(\frac{\pi}{2}\right) = 0, f''\left(\frac{\pi}{2}\right) < 0$
Local maximum
at $x = \frac{\pi}{2}$
y
1
$y = f(x) = \sin^2 x$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
x
$f'(-\pi) = 0, f''(-\pi) > 0$
Local minimum
at $x = -\pi$
$f'(0) = 0, f''(0) > 0$
Local minimum
at $x = 0$
$f'(\pi) = 0, f''(\pi) > 0$
Local minimum
at $x = \pi$

Figure 4.36


$y = f(x)$
f is differentiable on an interval
$\Longrightarrow f$ is a smooth curve
$y = f(x)$
$f' > 0 \Longrightarrow f$ is increasing
$y = f(x)$
$f' < 0 \Longrightarrow f$ is decreasing
+
−
or
−
+
f' changes sign $\Longrightarrow$
f has local maximum
or local minimum
$f' = 0$ and $f'' < 0 \Longrightarrow$
f has local maximum
$f' = 0$ and $f'' > 0 \Longrightarrow$
f has local minimum
or
$f'' > 0 \Longrightarrow f$ is concave up
or
$f'' < 0 \Longrightarrow f$ is concave down
+
−
f'' changes sign $\Longrightarrow$
f has inflection point