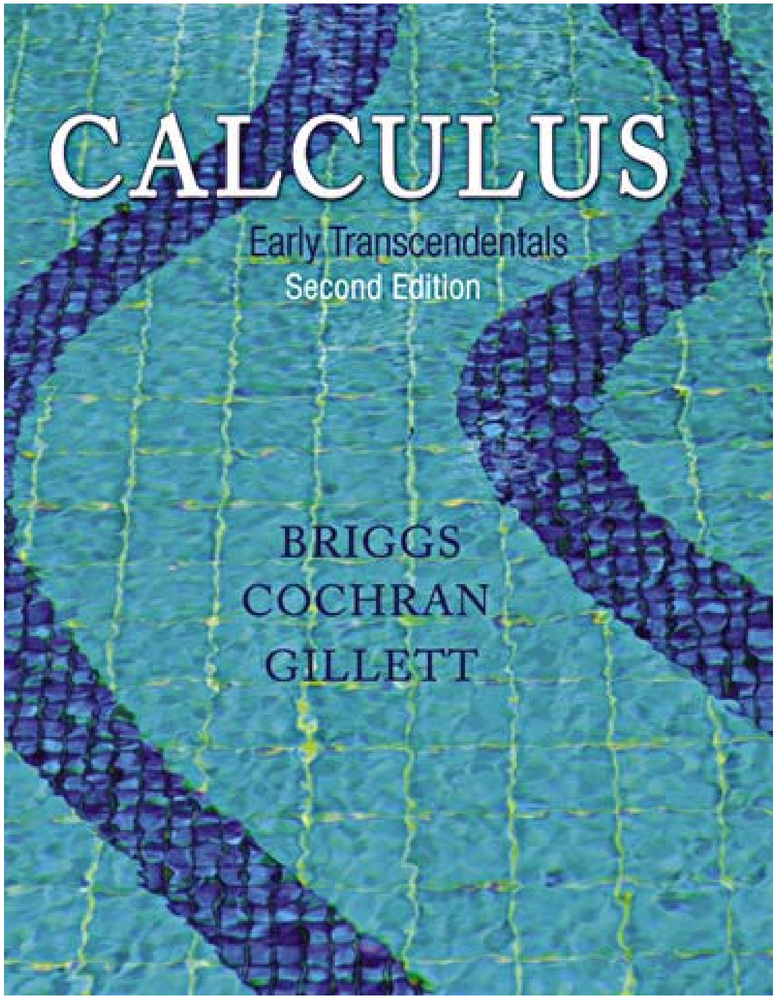
Chapter 8

Sequences and Infinite Series



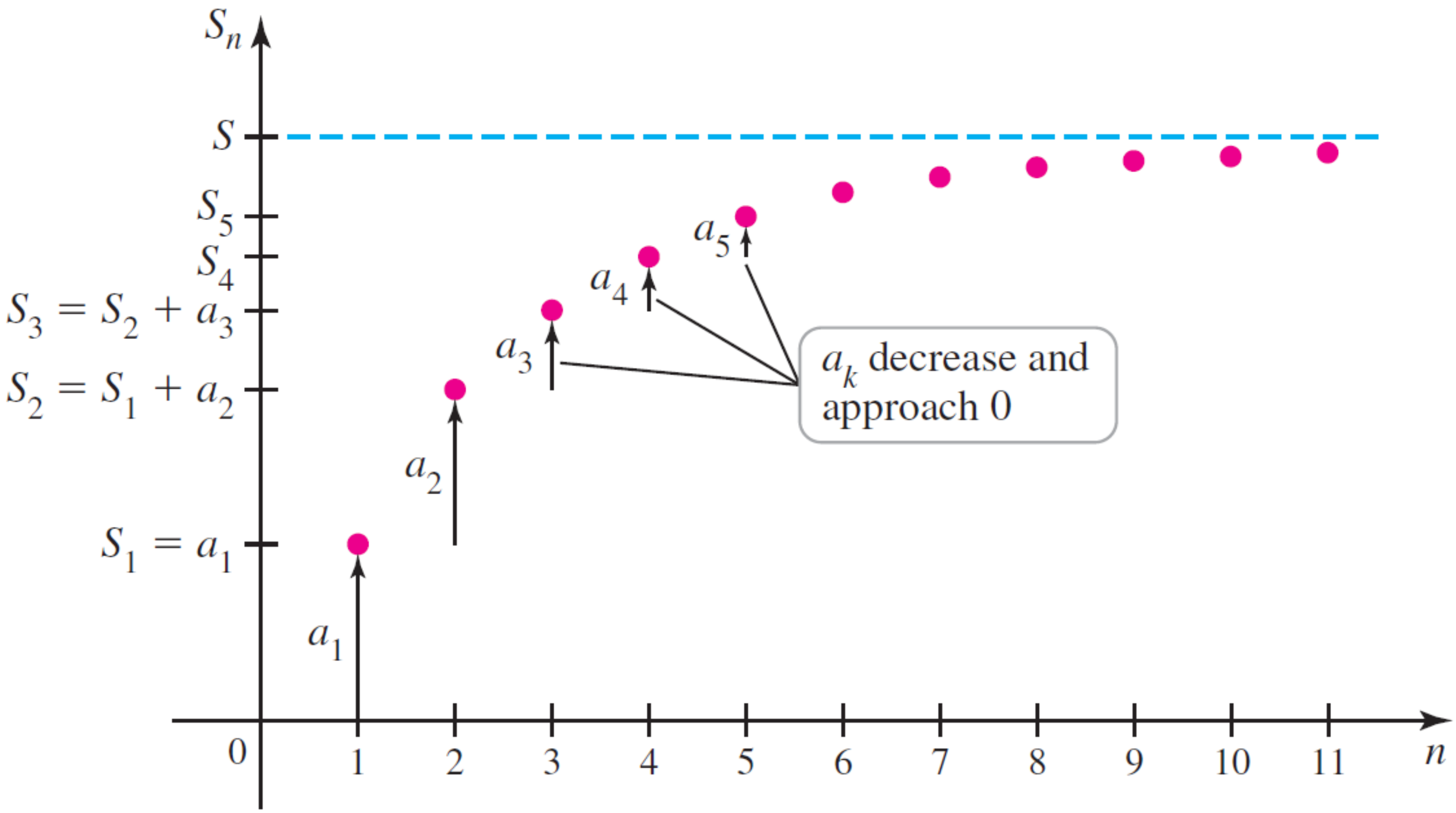
8.4

The Divergence and Integral Tests

THEOREM 8.8 Divergence Test

If $\sum a_k$ converges, then $\lim_{k \to \infty} a_k = 0$. Equivalently, if $\lim_{k \to \infty} a_k \neq 0$, then the series diverges.

Figure 8.25



$$\sum a_k \text{ converges} \Longrightarrow \lim_{n \to \infty} S_n = S \Longrightarrow \lim_{k \to \infty} a_k = 0$$

Table 8.3

n	S_n	n	S_n
10^3	≈ 7.49	10^{10}	≈ 23.60
10^4	≈ 9.79	10^{20}	≈ 46.63
10^5	≈ 12.09	10^{30}	≈ 69.65
10^6	≈ 14.39	10^{40}	≈ 92.68

Figure 8.26

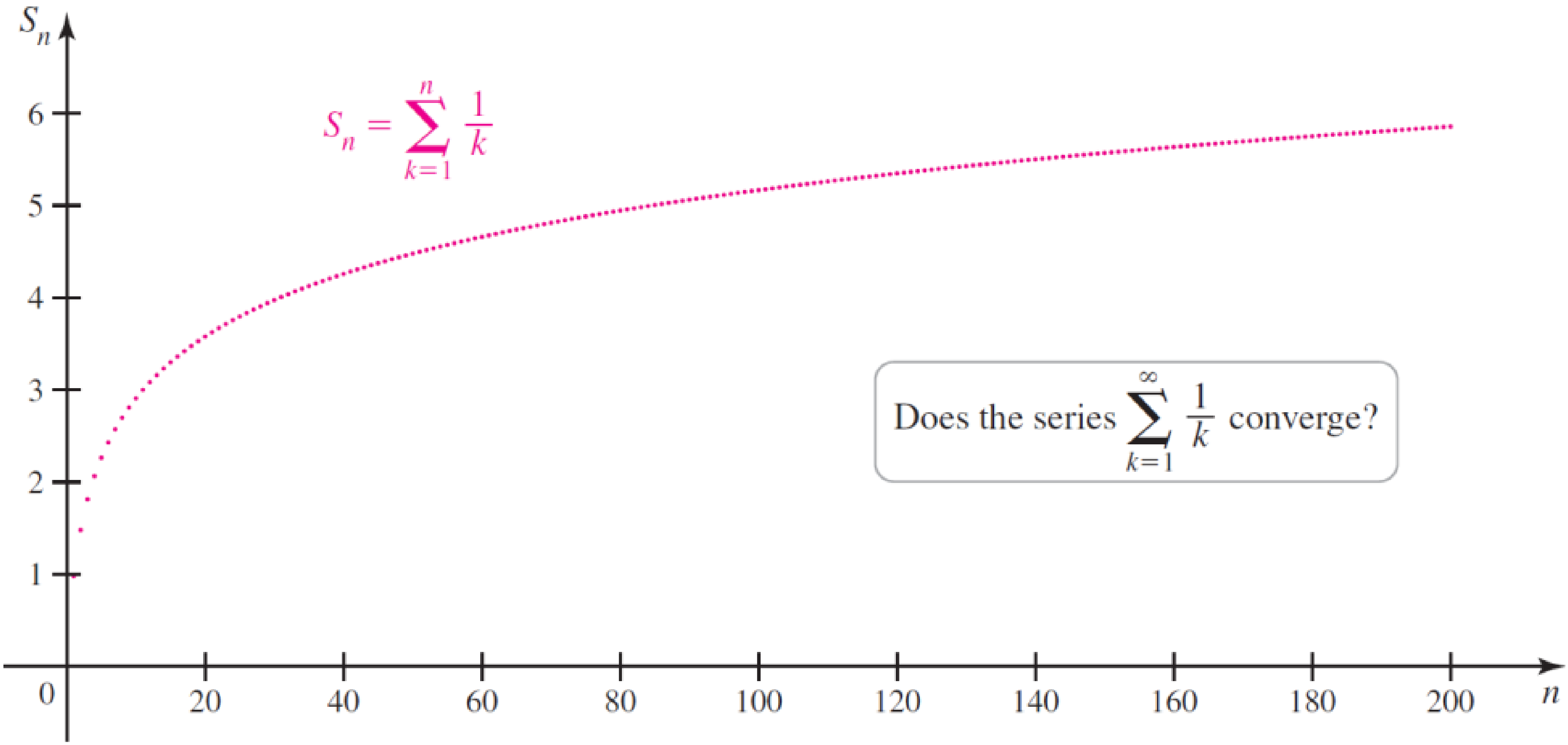
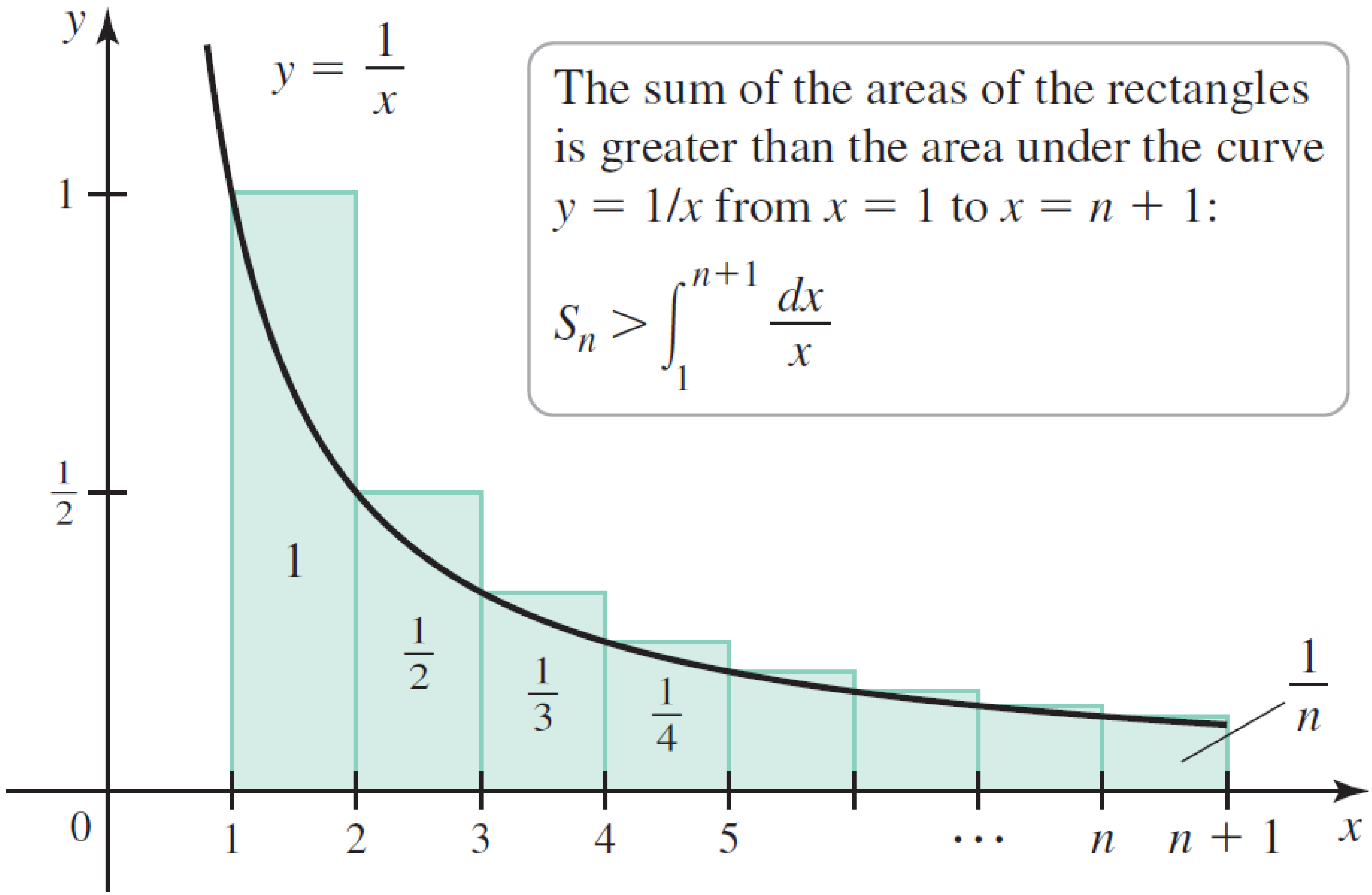

S_n
6
5
4
3
2
1
0
20
40
60
80
100
120
140
160
180
200
n
$S_n = \sum_{k=1}^{n} \frac{1}{k}$
Does the series $\sum_{k=1}^{\infty} \frac{1}{k}$ converge?

Figure 8.27



y
$y = \frac{1}{x}$
1
$\frac{1}{2}$
0
1
$\frac{1}{2}$
$\frac{1}{3}$
$\frac{1}{4}$
$\frac{1}{n}$
1
2
3
4
5
. . .
n
n + 1
x
The sum of the areas of the rectangles is greater than the area under the curve $y = 1/x$ from $x = 1$ to $x = n + 1$:
$S_n > \int_1^{n+1} \frac{dx}{x}$

THEOREM 8.9 Harmonic Series

The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ diverges—even though the terms of the series approach zero.

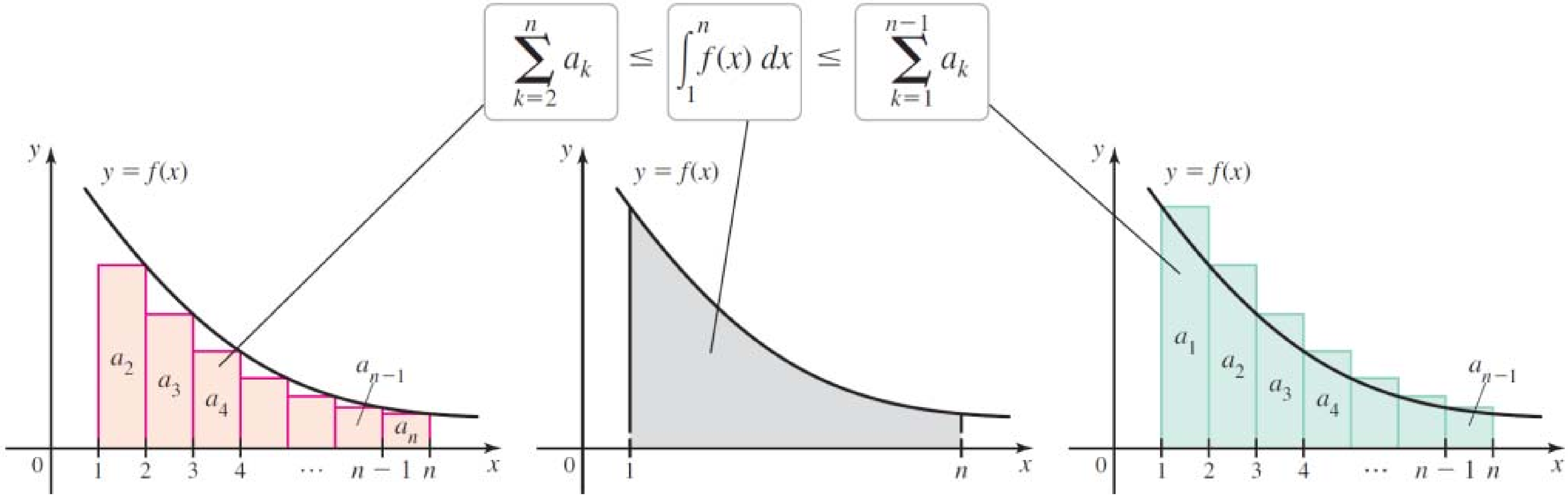
THEOREM 8.10 Integral Test

Suppose f is a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Then

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge. In the case of convergence, the value of the integral is *not* equal to the value of the series.

Figure 8.28

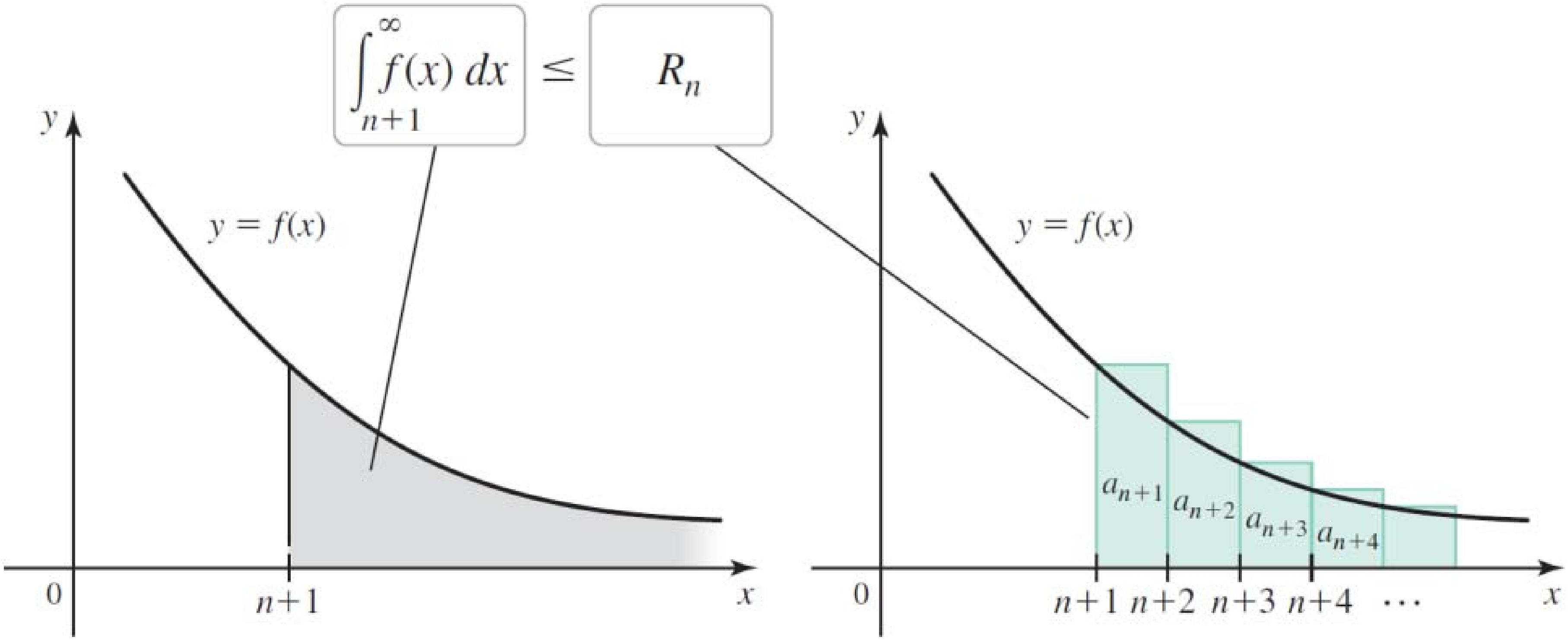


$\sum_{k=2}^{n} a_k \le \int_1^n f(x)\,dx \le \sum_{k=1}^{n-1} a_k$
y
y = f(x)
a_2
a_3
a_4
a_{n-1}
a_n
0
1
2
3
4
...
n − 1
n
x
y
y = f(x)
0
1
n
x
y
y = f(x)
a_1
a_2
a_3
a_4
a_{n-1}
0
1
2
3
4
...
n − 1
n
x

THEOREM 8.11 Convergence of the p-Series

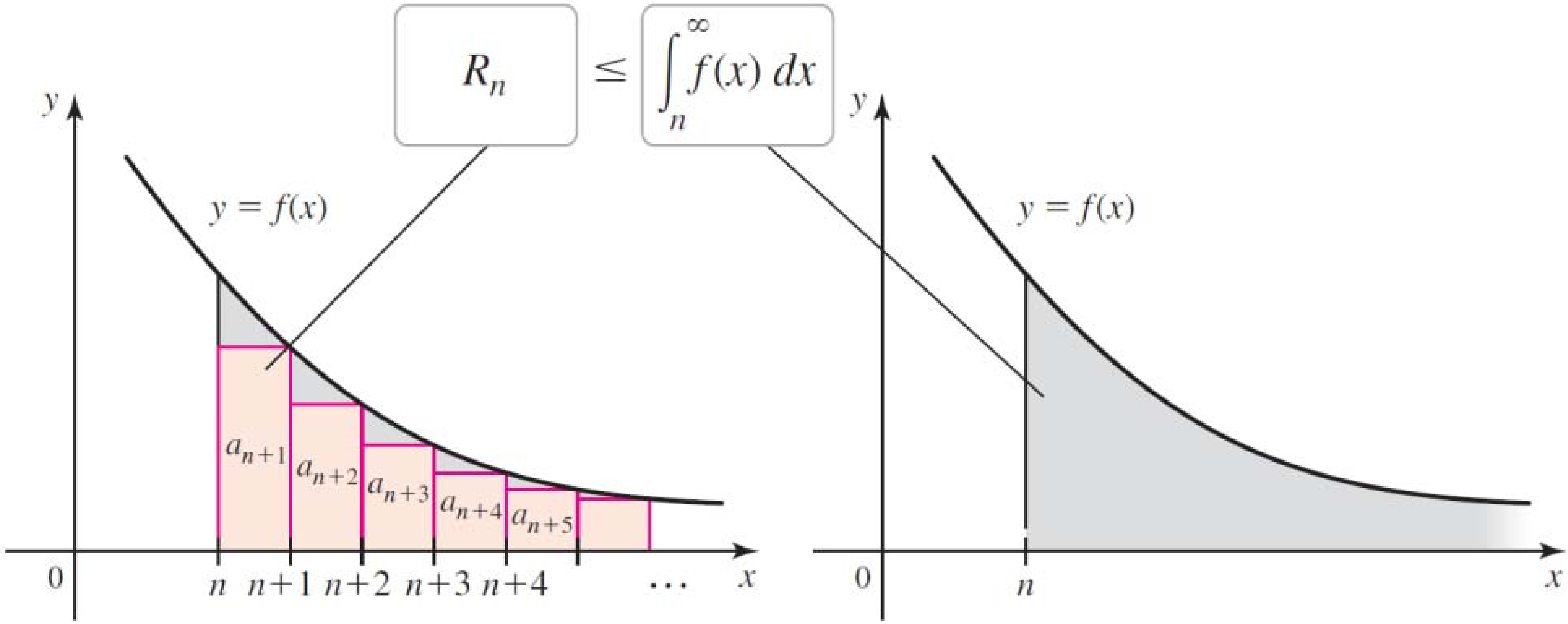
The ***p*-series** $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Figure 8.29



$\int_{n+1}^{\infty} f(x)\,dx$
≤
R_n
y
y = f(x)
0
n+1
x
y
y = f(x)
a_{n+1}
a_{n+2}
a_{n+3}
a_{n+4}
0
n+1 n+2 n+3 n+4 ...
x

Figure 8.30


R_n
$\leq$
$\int_n^\infty f(x)\,dx$
y
y = f(x)
a_{n+1}
a_{n+2}
a_{n+3}
a_{n+4}
a_{n+5}
0
n n+1 n+2 n+3 n+4 . . .
x
y
y = f(x)
0
n
x

THEOREM 8.12 Estimating Series with Positive Terms

Let f be a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergent series and let $S_n = \sum_{k=1}^{n} a_k$ be the sum of the first n terms of the series. The remainder $R_n = S - S_n$ satisfies

$$R_n < \int_n^{\infty} f(x)\,dx.$$

Furthermore, the exact value of the series is bounded as follows:

$$S_n + \int_{n+1}^{\infty} f(x)\,dx < \sum_{k=1}^{\infty} a_k < S_n + \int_n^{\infty} f(x)\,dx.$$

Figure 8.31 (a)

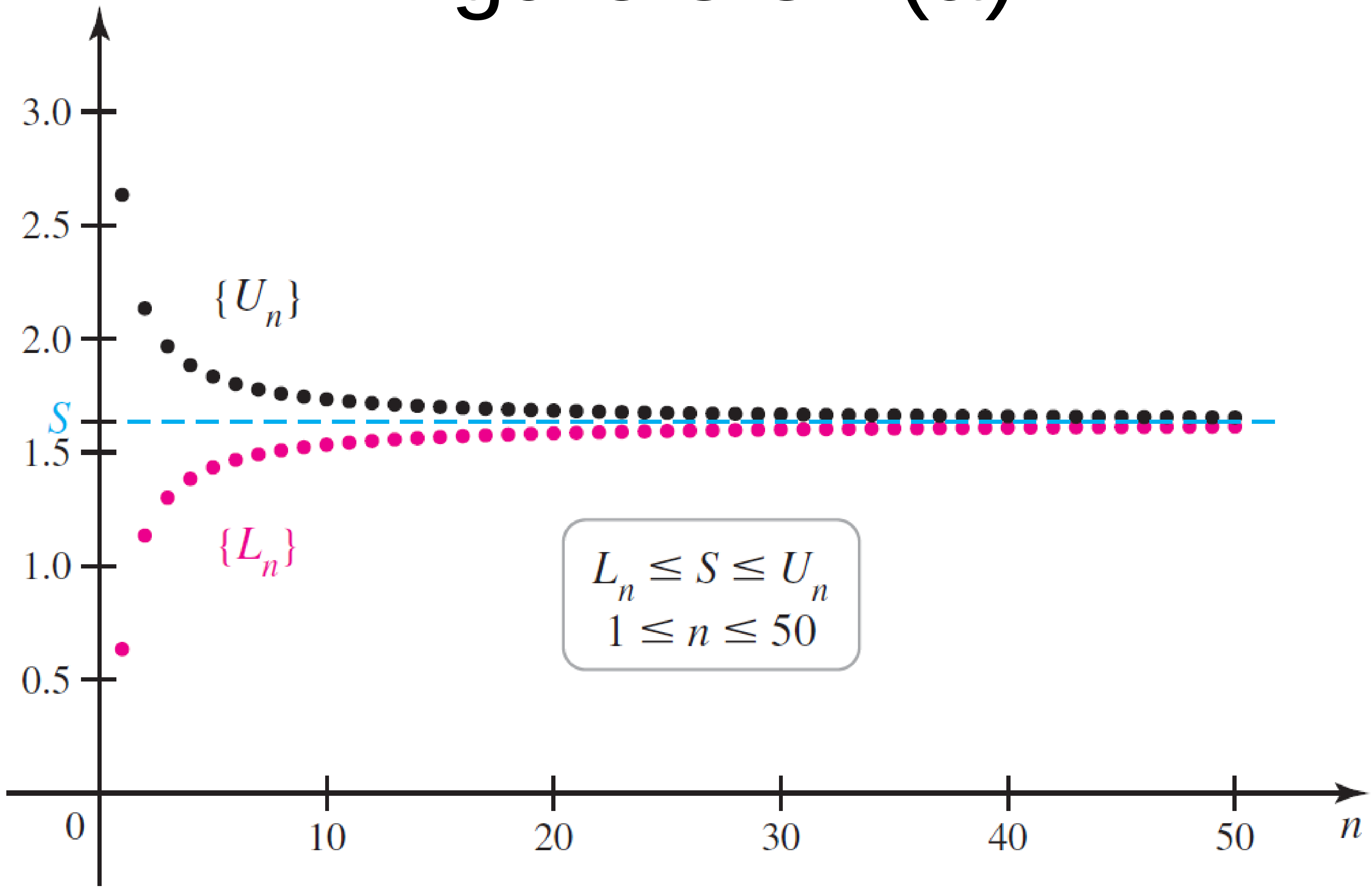
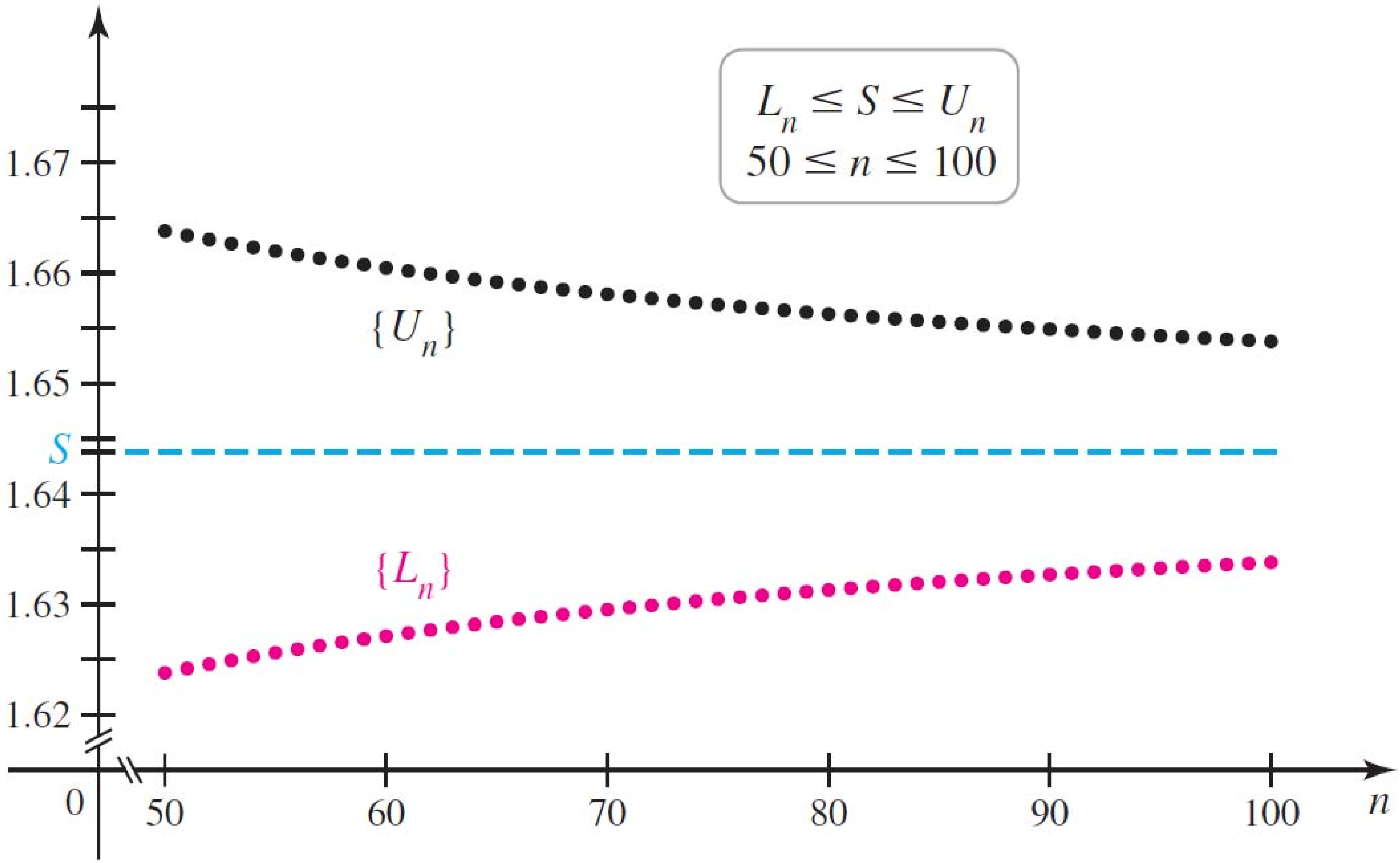


Figure 8.31 (b)



$L_n \le S \le U_n$
$50 \le n \le 100$
1.67
1.66
1.65
S
1.64
1.63
1.62
0
$\{U_n\}$
$\{L_n\}$
50
60
70
80
90
100
n

THEOREM 8.13 **Properties of Convergent Series**

1. Suppose $\sum a_k$ converges to A and c is a real number. The series $\sum ca_k$ converges, and $\sum ca_k = c\sum a_k = cA$.
2. Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum(a_k \pm b_k)$ converges, and $\sum(a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$.
3. If M is a positive integer, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ either both converge or both diverge. In general, *whether* a series converges does not depend on a finite number of terms added to or removed from the series. However, the *value* of a convergent series does change if nonzero terms are added or removed.