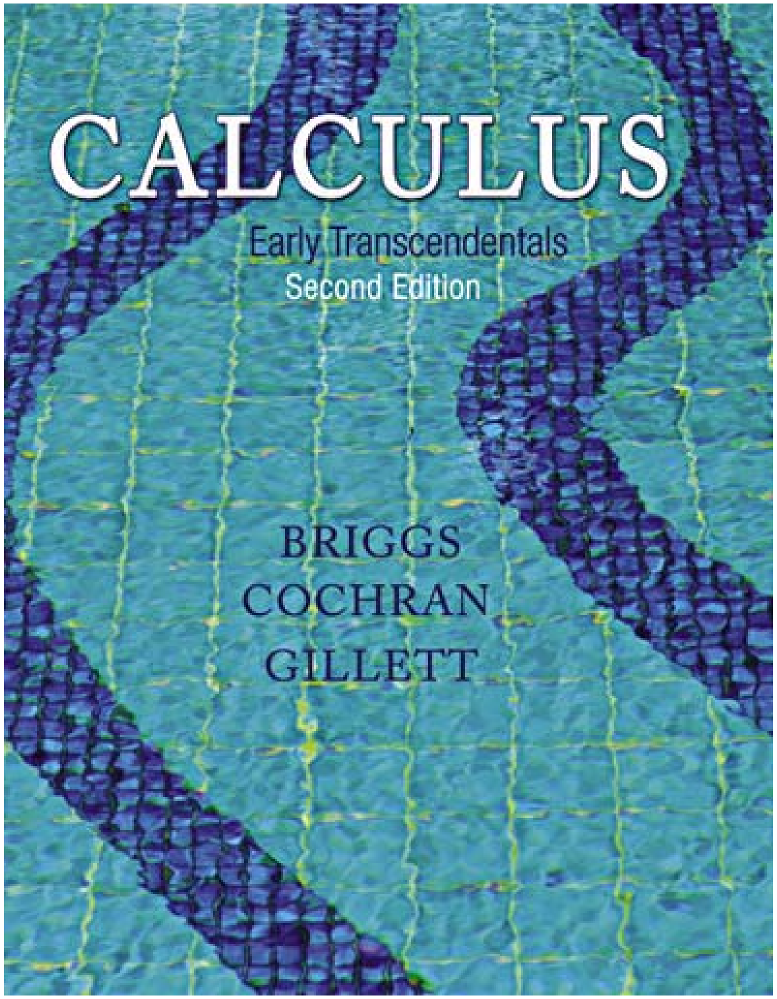
Chapter 8

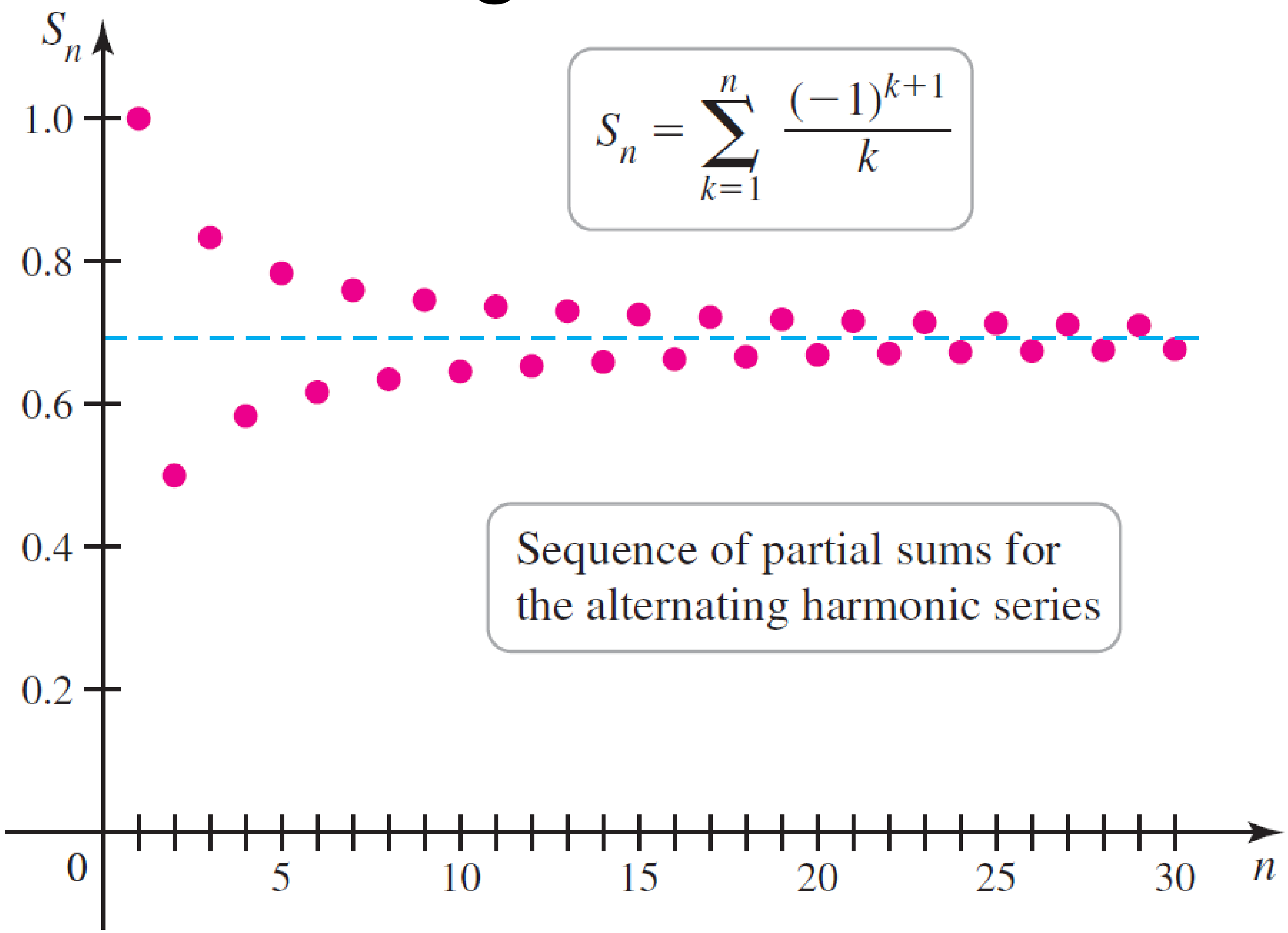
Sequences and Infinite Series



8.6

Alternating Series

Figure 8.34

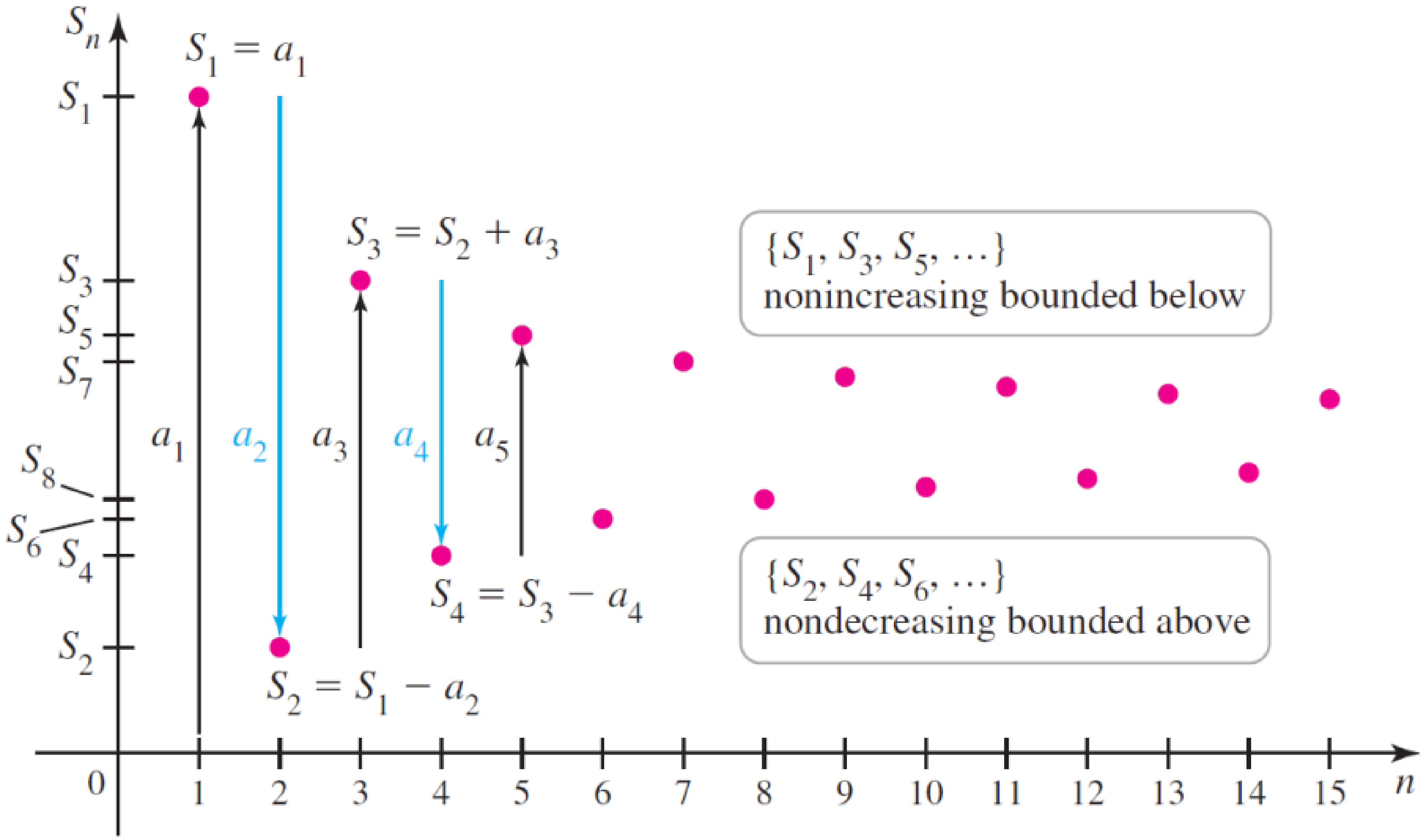

S_n
1.0
0.8
0.6
0.4
0.2
0
5
10
15
20
25
30
n
$S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$
Sequence of partial sums for the alternating harmonic series

THEOREM 8.18 Alternating Series Test

The alternating series $\sum(-1)^{k+1}a_k$ converges provided

1. the terms of the series are nonincreasing in magnitude ($0 < a_{k+1} \leq a_k$, for k greater than some index N) and
2. $\lim_{k\to\infty} a_k = 0$.

Figure 8.35



S_n
$S_1 = a_1$
S_1
S_3
S_5
S_7
S_8
S_6
S_4
S_2
a_1
a_2
a_3
a_4
a_5
$S_3 = S_2 + a_3$
$S_4 = S_3 - a_4$
$S_2 = S_1 - a_2$
$\{S_1, S_3, S_5, \ldots\}$
nonincreasing bounded below
$\{S_2, S_4, S_6, \ldots\}$
nondecreasing bounded above
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
n

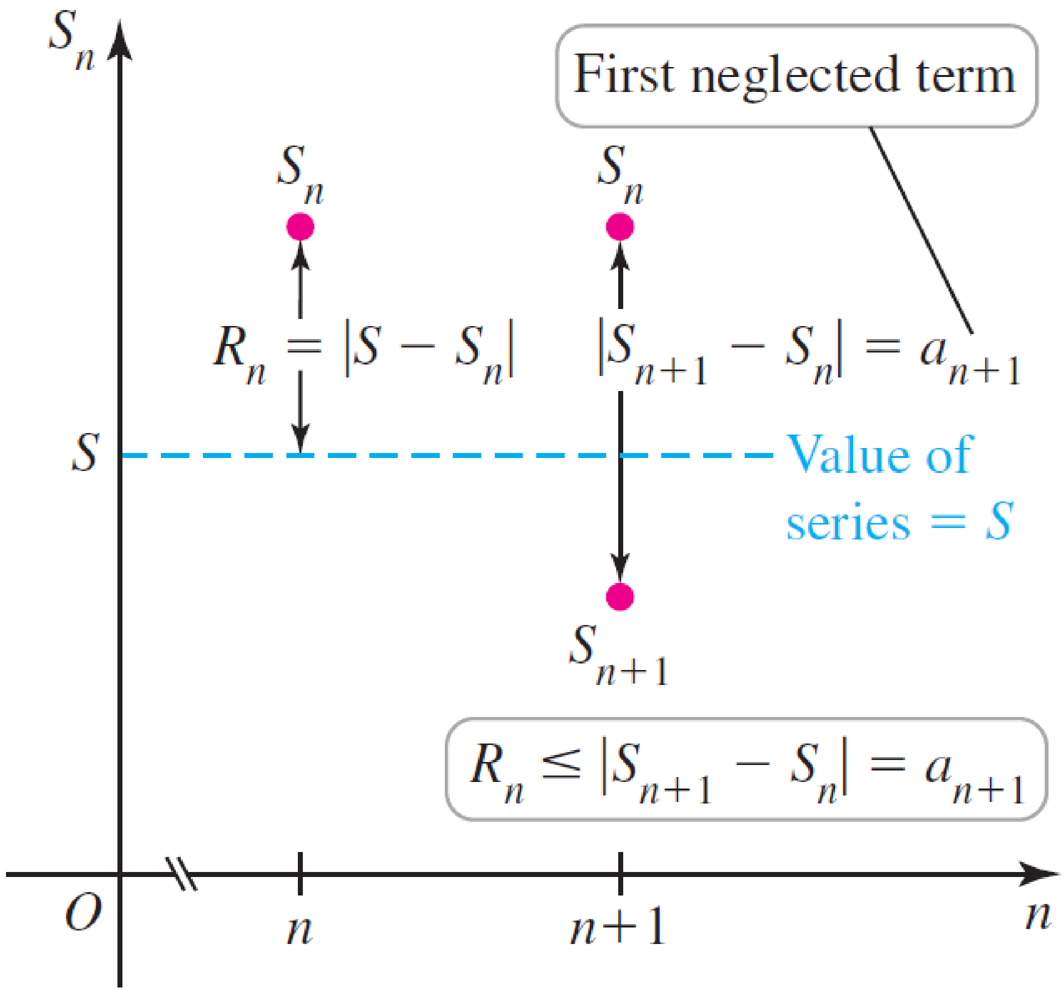
THEOREM 8.19 Alternating Harmonic Series

The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

converges (even though the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

diverges).

Figure 8.36



S_n
First neglected term
S_n
S_n
$R_n = |S - S_n|$
$|S_{n+1} - S_n| = a_{n+1}$
S
Value of
series = S
S_{n+1}
$R_n \leq |S_{n+1} - S_n| = a_{n+1}$
O
n
$n+1$
n

THEOREM 8.20 Remainder in Alternating Series

Let $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ be a convergent alternating series with terms that are nonincreasing in magnitude. Let $R_n = S - S_n$ be the remainder in approximating the value of that series by the sum of its first n terms. Then $|R_n| \leq a_{n+1}$. In other words, the magnitude of the remainder is less than or equal to the magnitude of the first neglected term.

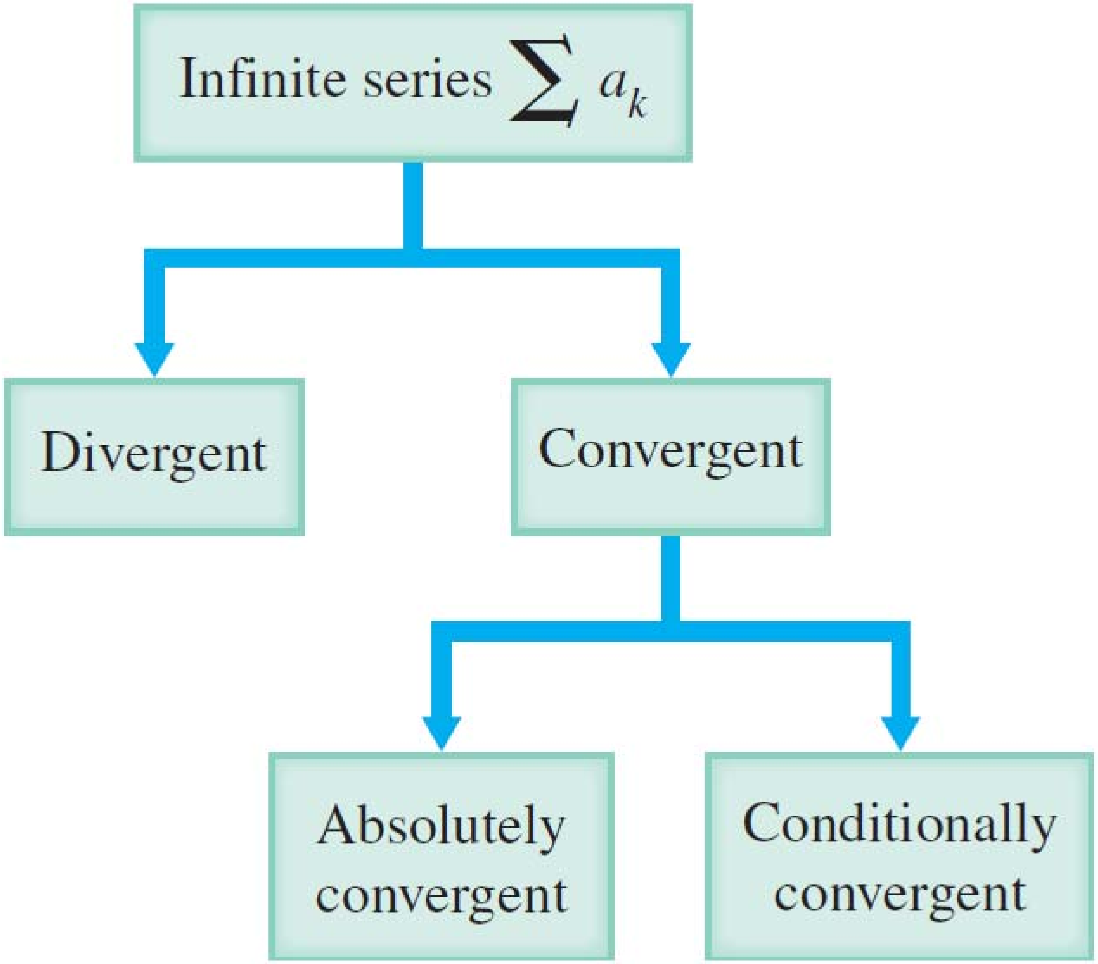
DEFINITION **Absolute and Conditional Convergence**

If $\sum |a_k|$ converges, then $\sum a_k$ **converges absolutely**. If $\sum |a_k|$ diverges and $\sum a_k$ converges, then $\sum a_k$ **converges conditionally**.

THEOREM 8.21 Absolute Convergence Implies Convergence

If $\sum |a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence). Equivalently, if $\sum a_k$ diverges, then $\sum |a_k|$ diverges.

Figure 8.37



Infinite series $\sum a_k$
Divergent
Convergent
Absolutely convergent
Conditionally convergent

Table 8.4 (1 of 2)

Series or Test	**Form of Series**	**Condition for Convergence**	**Condition for Divergence**	**Comments**
Geometric series	$\sum_{k=0}^{\infty} ar^k, a \neq 0$	$\lvert r \rvert < 1$	$\lvert r \rvert \geq 1$	If $\lvert r \rvert < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k\to\infty} a_k \neq 0$	Cannot be used to prove convergence
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing	$\int_1^{\infty} f(x)\, dx$ converges.	$\int_1^{\infty} f(x)\, dx$ diverges.	The value of the integral is not the value of the series.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests
Ratio Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$	$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} > 1$	Inconclusive if $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$, where $a_k \geq 0$	$\lim_{k\to\infty} \sqrt[k]{a_k} < 1$	$\lim_{k\to\infty} \sqrt[k]{a_k} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{a_k} = 1$

Table 8.4 (2 of 2)

Series or Test	Form of Series	Condition for Convergence	Condition for Divergence	Comments
Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$0 < a_k \le b_k$ and $\sum_{k=1}^{\infty} b_k$ converges.	$0 < b_k \le a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0, b_k > 0$	$0 \le \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0, 0 < a_{k+1} \le a_k$	$\lim_{k\to\infty} a_k = 0$	$\lim_{k\to\infty} a_k \ne 0$	Remainder R_n satisfies $\lvert R_n \rvert \le a_{n+1}$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k$, a_k arbitrary	$\sum_{k=1}^{\infty} \lvert a_k \rvert$ converges		Applies to arbitrary series