MATH 141

MIDTERM EXAM I ANSWERS

October 12, 2001

1. (10 pts) Find all x such that:

(a) $\left|\frac{x+5}{-3}\right| \geq 6$

Either $\frac{x+5}{-3} \geq 6$, or $\frac{x+5}{-3} \leq -6$. The first inequality simplifies as follows:

$$\frac{x+5}{-3} \geq 6 \implies x+5 \leq -18 \implies x \leq -23.$$

The second inequality simplifies as follows:

$$\frac{x+5}{-3} \leq -6 \implies x+5 \geq 18 \implies x \geq 13.$$

Thus $x \in (-\infty, -23] \cup [13, \infty)$.

(b) $|10x - 4| = 9$

Either $10x - 4 = 9$, or $-(10x-4) = 9$. The first case gives $x = \frac{13}{10}$, and the second case gives $x = -\frac{1}{2}$.

2. (12 pts) Given $P(-1, 2)$ and $Q(3, 5)$, find the following.

(a) The distance $|PQ|$.

The distance formula says that $D = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$. Applying this here, we get

$$D = \sqrt{(5-2)^2 + (3-(-1))^2} = \sqrt{9+16} = \sqrt{25} = 5.$$

(b) The slope of the line through P and Q.

The slope is $\frac{y_2 - y_1}{x_2 - x_1}$. Applying this here, we get

$$\frac{5-2}{3-(-1)} = \frac{3}{4}.$$

(c) The line through P parallel to the x-axis.

A line parallel to the x-axis has slope 0. The y-coordinate of P is 2, so the horizontal line through this point is

$$y = 2.$$

(d) The line through Q perpendicular to the line $2x + 6y + 3 = 0$.

Let's put the line $2x + 6y + 3 = 0$ into slope-intercept form:

$$6y = -2x - 3 \implies y = -\frac{2}{6}x - \frac{3}{6} \implies y = -\frac{1}{3}x - \frac{1}{2}.$$

Thus this line has slope $-\frac{1}{3}$. Any line perpendicular to this line has slope 3 (take the negative reciprocal of $-1/3$).

So we need to find the line through $Q(3, 5)$ which has slope 3:

$$\frac{y-5}{x-3} = 3 \implies y - 5 = 3(x - 3) \implies y = 3x - 4.$$

This is the line we want.

3. (4 pts) Do the following.

(a) Convert the angle of $\frac{-8\pi}{5}$ radians to degrees.

$\frac{-8\pi}{5}$ radians is $\frac{-8*180}{5}$ degrees. This simplifies to -288 degrees.

(b) Convert the angle of 150 degrees to radians.

150 degrees $= 150\frac{\pi}{180}$ radians. This simplifies to $\frac{5\pi}{6}$ radians.

4. (10 pts) Find the exact values.

(a) $\sin\left(\frac{4\pi}{8}\right)$

$$= \sin(\frac{\pi}{2}) = 1.$$

(b) $\tan\left(\frac{-\pi}{3}\right)$

$$= \frac{\sin(-\pi/3)}{\cos(-\pi/3)} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}.$$

(c) $\cos^2\left(\arctan\left(\frac{1}{8}\right)\right) + \sin^2\left(\arctan\left(\frac{1}{8}\right)\right)$

$\cos^2\theta + \sin^2\theta = 1$ for any angle θ.

(d) $2\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$

For any θ, $2\sin\theta\cos\theta = \sin(2\theta)$. Thus

$$2\sin(\pi/12)\cos(\pi/12) = \sin(2\frac{\pi}{12}) = \sin(\frac{\pi}{6}) = \frac{1}{2}.$$

(e) $\arccos\left(\cos\left(\frac{\pi}{9}\right)\right)$

By definition, $\arccos x$ is the angle in $[0, \pi]$ whose cosine is x. Thus $\arccos(\cos(\frac{\pi}{9}))$ is the angle in $[0, \pi]$ whose cosine is $\cos(\frac{\pi}{9})$. This angle is $\frac{\pi}{9}$.

Aside: In the above example, the arccos and cos cancel. However, this does not always happen. Note for example that $\arccos(\cos(-\frac{\pi}{2}))$ is the angle in $[0, \pi]$ whose cosine is $\cos(-\frac{\pi}{2}) = 0$. Because $-\frac{\pi}{2}$ doesn't live in $[0, \pi]$, we need to look for another angle whose cosine is 0, namely $\frac{\pi}{2}$. So $\arccos(\cos(-\frac{\pi}{2})) = \frac{\pi}{2}$.

5. (9 pts) Find functions whose graphs are obtained from the graph of $f(x) = x^3 + 1$ by:

(a) Shifting 6 units to the right.

$y = (x - 6)^3 + 1$

(b) Vertically stretching by a factor of 10.

$y = 10(x^3 + 1)$

(c) Reflecting about the y-axis and then shifting 4 units up.

First reflect about the y-axis:

$$y = (-x)^3 + 1 = -x^3 + 1.$$

Now shift the above result up by 4:

$$y = -x^3 + 1 + 4 = -x^3 + 5.$$

6. (10 pts) Do the following.

(a) Let $f(x) = \ln x$ and $g(x) = 1 - x^2$. Find $f \circ g$ and $g \circ f$ and their domains.

$$f \circ g(x) = f(g(x)) = f(1 - x^2) = \ln(1 - x^2).$$

To find the domain of the above function, note that ln must take a positive input. So we need

$$1 - x^2 > 0,$$

or

$$(1 - x)(1 + x) > 0.$$

The roots are $x = 1$, and $x = -1$, so there are three intervals to consider: $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. We just have to plug in a test number from each interval to see which intervals to include in the solution. If we plug in $x = 2$, we get a negative number. The same happens for $x = -2$. Thus we can discard the intervals $(-\infty, -1)$ and $(1, \infty)$ from consideration. Plugging in the test number $x = 0$, we get 1, which is positive. Hence the solution set of the inequality is the open interval $x \in (-1, 1)$. This is the domain of $f \circ g$. (Equivalently, $-1 < x < 1$.)

Now we solve $g \circ f$:

$$g \circ f(x) = g(f(x)) = g(\ln x) = 1 - (\ln x)^2.$$

In order to plug in a value for x here, it must be greater than zero, so the domain is $x \in (0, \infty)$, i.e. all $x > 0$.

(b) Find f and g such that $f \circ g = \tan\left(\sqrt[3]{x}\right)$.

There are lots of possible solutions here. The most natural one is $g(x) = \sqrt[3]{x}$ and $f(x) = \tan x$. Then

$$f \circ g(x) = f(g(x)) = f(\sqrt[3]{x}) = \tan(\sqrt[3]{x}).$$

Another possibility is to take $f(x) = \tan(\sqrt[3]{x})$ and $g(x) = x$. Or even $f(x) = x$ and $g(x) = \tan(\sqrt[3]{x})$. (Nobody did this on the test though!)

7. (10 pts) Find the inverse function for $f(x) = \dfrac{x+1}{2x+5}$.

Let $y = f^{-1}(x)$. Then $f(y) = x$, i.e.

$$\frac{y+1}{2y+5} = x.$$

We need to solve for y:

$$y + 1 = x(2y+5) = 2xy + 5x.$$

Putting all the y terms on the left, we get:

$$y - 2xy = 5x - 1.$$

Now factor y out of the left:

$$y(1-2x) = 5x - 1.$$

Dividing, we get

$$f^{-1}(x) = y = \frac{5x-1}{1-2x}.$$

8. (10 pts) Solve the equations given below.

(a) $5^x + 5^{x+1} = 10$
$5^x + 5^x 5 = 10$
$5^x(1+5) = 10$
$5^x = \frac{10}{6} = \frac{5}{3}$
$x = \log_5(\frac{5}{3})$.

(b) $\ln(x^2) - 9 = \ln x$
$\ln(x^2) - \ln x = 9$
$\ln \frac{x^2}{x} = 9$
$\ln x = 9$
$x = e^9$.

9. (10 pts) A ladybug is crawling on the y-axis. If her position at time t is given by $y = 2t^2 + 1$, find her average velocity between $t = 1$ and $t = 3$ seconds.

Average velocity = Average rate of change of position = $\frac{p(3)-p(1)}{3-1}$
$= \frac{(2(3)^2+1)-(2(1)^2+1)}{2} = \frac{16}{2} = 8$.

10. (15 pts) Evaluate each of the following limits. For infinite limits, specify ∞ or $-\infty$.

(a) $\lim_{x\to 3} \dfrac{x^2+1}{(x+2)(x+6)}$

$= \frac{10}{(5)(9)} = \frac{2}{9}$.

(b) $$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

$$= \lim_{x\to 0} \frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})} = \lim_{x\to 0} \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \lim_{x\to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}} = \frac{2}{1+1} = 1.$$

(c) $\lim_{x\to 0} |x| \cos\left(\dfrac{\pi}{x}\right)$

We know that $-1 \le \cos(\frac{\pi}{x}) \le 1$. We can multiply this through by $|x|$ without reversing inequalities since $|x| \ge 0$:

$$-|x| \le |x|\cos(\frac{\pi}{x}) \le |x|.$$

The function on the left and the function on the right both go to 0 as $x \to 0$. Therefore by the squeeze theorem,

$$\lim_{x\to 0} |x|\cos(\pi/x) = 0.$$

(d) $\lim_{x\to 5^-} \dfrac{4}{x-5}$ When $x < 5$ and x is very close to 5, this expression looks like

$$\frac{4}{\text{small negative}},$$

so in the limit, we get $-\infty$.

(e) $$\lim_{x\to 7} \frac{\frac{1}{x}-\frac{1}{7}}{x-7} = \lim_{x\to 7} \frac{\frac{7}{7x}-\frac{x}{7x}}{x-7} = \lim_{x\to 7} \frac{7-x}{7x}\,\frac{1}{x-7} = \lim_{x\to 7} \frac{-1}{7x} = -1/49.$$