Practice Final Exam (b) with answers

Math 142

PART I:

1. (10 points) Let $f(x) = 3x - x^3$. Find the absolute maximum and minimum values of $f(x)$ on the closed interval $[0, 2]$ and determine at which x-values they occur.

SOLUTION: The maximum is at $(1, 2)$ and the minimum is at $(2, -2)$.

2. (28 points) Throughout this problem we consider the function $f(x) = \dfrac{48(x-8)}{(x+4)^2}$. The first and second derivatives of f are

$$f'(x) = \frac{48(20-x)}{(x+4)^3} \qquad \text{and} \qquad f''(x) = \frac{96(x-32)}{(x+4)^4}.$$

2a. (2 points) At what point(s) does the graph $y = f(x)$ intersect the y-axis?

SOLUTION: At $y = -24$.

2b. (2 points) At what point(s) does the graph $y = f(x)$ intersect the x-axis?

SOLUTION: At $x = 8$.

2c. (2 points) What are the horizontal asymptotes of $y = f(x)$?

SOLUTION: $y = 0$.

2d. (2 points) What are the vertical asymptotes of $y = f(x)$?

SOLUTION: $x = -4$.

2e. (2 points) Find the critical point(s) of $y = f(x)$.

SOLUTION: $x = 20$.

2f. (3 points) On which intervals is $f(x)$ increasing?

SOLUTION: $4 < x < 20$.

2g. (3 points) On which intervals is $f(x)$ decreasing?

SOLUTION: $x < 4$ and $x > 20$.

2h. (2 points) Find the x-coordinates of the inflection point(s) of $f(x)$.

SOLUTION: $(32, 8/9)$.

2i. (3 points) On which intervals is $f(x)$ concave up?

SOLUTION: $x > 32$.

2j. (3 points) On which intervals is $f(x)$ concave down?

SOLUTION: $x < -4$ and $-4 < x < 32$.

2k. (4 points) Sketch the graph of $y = f(x)$ using the information from parts (a)–(j).

3. (18 points) (18 point question removed from this practice exam since it covers material from 141, not 142).

4. (12 points) A metal box (without top) is to be constructed from a square sheet of metal that is 10 inches on a side by first cutting square pieces of the same size from each of the corners and then folding up the sides. Find the dimensions of the box with the largest volume that can be constructed in this manner.

SOLUTION: Let x be the side of each square to be removed. Then the volume is

$$V(x) = x(10-x)^2 = 100x - 40x^2 + 4x^3$$

To maximize this function, we find when its derivative

$$V'(x) = 100 - 80x + 12x^2 = 4(-5+x)(-5+3x)$$

vanishes. This occurs when $x = 5$ (no volume) and $x = 5/3$, making the dimensions of the biggest box $5/3 \times 6\,2/3 \times 6\,2/3$ with volume $2000/27$ cubic inches.

5. (20 points) Evaluate the following integrals.

5a. (5 points) $\int \sin^4 x \cos x \, dx = \frac{\sin(x)}{8} - \frac{\sin(3\,x)}{16} + \frac{\sin(5\,x)}{80}$

5b. (5 points) $\displaystyle\int_0^1 \frac{x^5}{(x^6+1)^3}\,dx = \left.\frac{-1}{12\,(1+x^6)^2}\right|_0^1 = \frac{1}{16}.$

5c. (5 points) $\displaystyle\int_0^2 2xe^{3x}\,dx = 2\,e^{3\,x}\left(-\left(\frac{1}{9}\right)+\frac{x}{3}\right)\Bigg|_0^2 = 2\left(\frac{1}{9}+\frac{5\,e^6}{9}\right).$

5d. (5 points) $\displaystyle\int e^{2x}\sin x\,dx = \frac{-\left(e^{2\,x}\,\cos(x)\right)}{5}+\frac{2\,e^{2\,x}\,\sin(x)}{5}$

6. (12 points) Evaluate the following integrals.

6a. (6 points) $\displaystyle\int \frac{5x-3}{x^2-2x-3}\,dx = 3\,\log(-3+x)+2\,\log(1+x)$

6b. (6 points) $\displaystyle\int_0^\infty \frac{dx}{x^2+1} = \arctan(x)|_0^\infty = \frac{\pi}{2}$

PART II:

7. (12 points) Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

SOLUTION: The line and the parabola intersect at $(-1, 1)$ and $(2, -2)$, so we integrate from -1 t0 2.

$$A = \int_{-1}^{2} (2-x^2)-(-x)\,dx = 2\,x+\frac{x^2}{2}-\frac{x^3}{3}\Bigg|_{-1}^{2} = \frac{9}{2}.$$

8. (18 points) Consider the region bounded by the curve $y = \sqrt{x}$, the line $x = 4$, and the x-axis. Find the volume of the solid formed by rotating this region around the x-axis.

SOLUTION:

$$V = \int_0^4 \pi(\sqrt{x})^2\,dx = \frac{\pi x^2}{2}\Bigg|_0^4 = 8\pi.$$

9. (20 points) Consider the triangle bounded by the lines $y = x - 1$, $y = 0$, and $x = 2$. Rotate this triangle **around the y-axis**. What is the volume of the resulting solid?

SOLUTION:

$$V = \int_1^2 2\pi x(x-1)\,dx = 2\,\pi\left(\frac{-x^2}{2}+\frac{x^3}{3}\right) = \frac{5\,\pi}{3}.$$

10. (15 points)

10a. (5 points) Write down an integral which gives the value of the arclength of the parabola $y = x^2 + 2x + 4$ from $x = 0$ to $x = 2$. **Do not evaluate the integral.**

$$\int_0^2 \sqrt{1 + (2 + 2\,x)^2}\,dx$$

10b. (10 points) A car is travelling down a road in such a way that its velocity at time t hours is $v(t) = 30 + 15t^2$ miles per hour. Find the car's average velocity over the time interval from $t = 0$ to $t = 2$.

$$v = \frac{1}{2}\int_0^2 30 + 15t^2\,dt = 50.$$

11. (15 points) Suppose the work required to stretch a certain spring 1 meter beyond its natural length is 10 joules. Find the work (in joules) required to stretch it from 1 meter beyond its natural length to 3 meters beyond.

SOLUTION: Let k be the constant associated with the spring. Then the work need to stretch it 1m beyond its natural length is

$$\int_0^1 kx\,dx = \frac{k}{2} = 10$$

so $k = 20$. Thus the work required to stretch it from 1m to 3m beyond its natural length is

$$\int_1^3 20x\,dx = 10x^2\Big|_1^3 = 80 \text{ joules.}$$

12. (20 points) A swimming pool has the shape of a cylinder with (horizontal) radius 5 meters and depth 3 meters. Assume that the pool contains water (density 1000 kg/m^3) to a depth of 2 meters. Find the work required to pump all of the water out over the top of the pool. (The acceleration of gravity is $g = 9.8$ m/s^2.)

SOLUTION:

$$W = 9800\int_0^3 25\pi x\,dx = 1102500\pi \text{ joules.}$$