MATH 142

FINAL EXAM Answer key
May 7, 2001

Part I

1. (20 points) Evaluate each of the following integrals.

(a) $\displaystyle\int x^2 \cos(x^3)\, dx = \frac{\sin(x^3)}{3} + c$

(b) $\displaystyle\int \sin^3 x \ \cos^3 x\, dx = \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + c_1 = -\frac{\cos^4 x}{4} + \frac{\cos^6 x}{6} + c_2$

(c) $\displaystyle\int_1^2 x^3 e^{x^4}\, dx = \frac{e^{16} - e}{4}$

(d) $\displaystyle\int (x+1)(x^2+2x+5)^{10}\, dx = \frac{(x^2+2x+5)^{11}}{22} + c$

2. (10 points) The temperature on a beautiful spring day in Rochester (in degrees Fahrenheit) is

$$T(t) = 60 + 20\sin\left(\frac{2\pi t}{24}\right)$$

where t is the number of hours after 9am. Find the average temperature T between 9 am and 3 pm.

Solution:

$$T = \frac{1}{6}\int_0^6 T(t)\, dt = 60 + \frac{40}{\pi}.$$

3. (15 points) Let R be the region bounded by $y = 2x^2$, $y = x^2 + 1$, and the y-axis.

(a) Find the area of R.

SOLUTION:

$$A = \int_0^1 x^2 + 1 - 2x^2 \, dx = \frac{2}{3}.$$

(b) Find the volume of the solid obtained by rotating R about the x-axis. (The washer method is recommended.)

SOLUTION:

$$V = \pi \int_0^1 (x^2 + 1)^2 - 4x^4 \, dx = \frac{16\,\pi}{15}.$$

(c) Find the volume of the solid obtained by rotating R about the y-axis. (The shell method is recommended.)

SOLUTION:

$$V = 2\pi \int_0^1 x(x^2 + 1 - 2x^2) \, dx = \frac{\pi}{2}.$$

4. (10 points) A force of 2N is required to keep a spring 0.5m from its natural position. How many Joules of work are needed to stretch it 1m from its natural position?

SOLUTION: The constant k associated with this spring is $2/.5 = 4N/m$, so

$$W = \int_0^1 4x \, dx = 2 \text{ Joules.}$$

5. (10 points) A particle has acceleration $a(t) = 3t$.

(a) Find its velocity $v(t)$ assuming that $v(0) = 0$. (Recall that $a = \frac{dv(t)}{dt}$.)

SOLUTION:

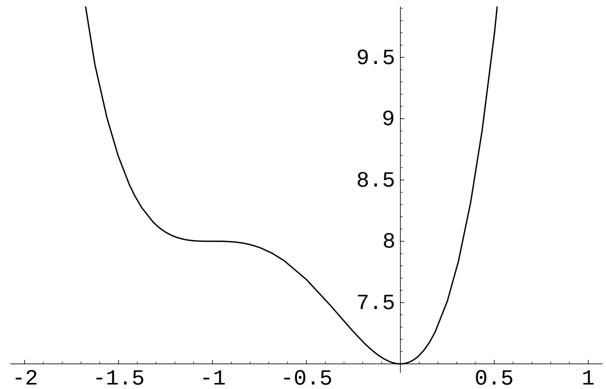
$$v(t) = \int a(t) \, dt = \frac{3t^2}{2}$$

(b) Find its displacement over the interval $0 \leq t \leq 5$.

SOLUTION:

$$x = \int_0^5 v(t) \, dt = \frac{125}{2}.$$

6. (21 points) Let $f(x) = 3x^4 + 8x^3 + 6x^2 + 7$, so $f'(x) = 12x(x+1)^2$ and $f''(x) = 12(3x+1)(x+1)$.



(a) Find the interval(s) where $f(x)$ is increasing.

SOLUTION: $x > 0$

(b) Find the interval(s) where $f(x)$ is decreasing.

SOLUTION: $x < -1$ and $-1 < x < 0$.

(c) Find the local maxima of $f(x)$.

SOLUTION: None.

(d) Find the local minima of $f(x)$.

SOLUTION: $x = 0$.

(e) Find the point(s) of inflection of $f(x)$.

SOLUTION: $x = -1$ and $x = -1/3$.

(f) Find the interval(s) where $f(x)$ is concave upward.

SOLUTION: $x < -1 < x$ and $x > -1/3$.

(g) Find the interval(s) where $f(x)$ is concave downward.

SOLUTION: $-1 < x < -1/3$.

Part II

7. (15 points) Evaluate the following integrals.

(a) $\int x^3 \ln(x)\, dx = \frac{x^4\,(-1+4\log(x))}{16} + c$

(b) $\int_0^1 \frac{1}{x^2-5x+6}\, dx = \log(-3+x) - \log(-2+x) + c$

(c) $\int_0^\infty \frac{1}{1+x^2}\, dx = \frac{\pi}{2}$

8. (10 points) A function $f(x)$ is known to have the following values.

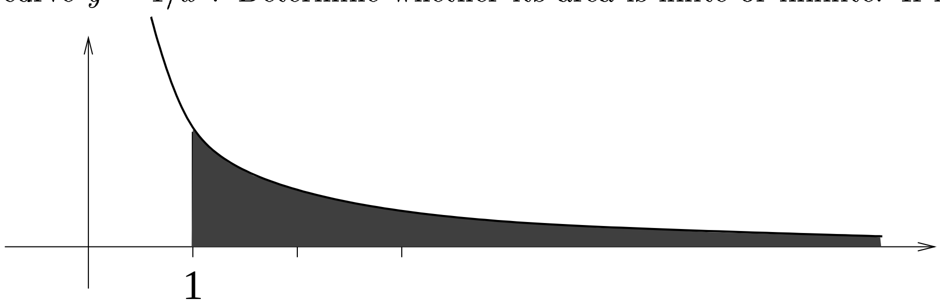
x	0	2	4	6	8
$f(x)$	1	4	5	5	7

Use Simpson's rule to approximate $\int_0^8 f(x)\, dx$.

SOLUTION:

$$\frac{\Delta x}{3}\left(f(0)+4f(2)+2f(4)+4f(6)+f(8)\right) = \frac{2}{3}(1+16+10+20+7) = 36.$$

9. (10 points) Consider the region bounded by the x-axis, the line $x = 1$ and the curve $y = 1/x^2$. Determine whether its area is finite or infinite. If it is finite, calculate it.



SOLUTION:

$$\int_1^\infty \frac{dx}{x^2} = 1.$$

10. (20 points) Evaluate the following integrals.

(a) $\displaystyle\int \frac{1}{(x^2+4)^{3/2}}\,dx = \frac{x}{4\sqrt{4+x^2}} + c$

(b) $\displaystyle\int e^{-x}\sin x\,dx = \frac{-e^{-x}(\cos x + \sin x)}{2} + c$

(c) $\displaystyle\int \frac{x+2}{x\,(x^2+1)}\,dx = \arctan(x) + 2\log(x) - \log(1+x^2) + c$

11. (10 points) Consider the ellipse defined by the equation $4x^2 + y^2 = 1$. Set up (but do not evaluate) the integral needed to find the arc length of the upper half of it for $-1/2 \le x \le 1/2$.

SOLUTION: Let $f(x) = \sqrt{1-4x^2}$. Then

$$s = \int_{-1/2}^{1/2} \sqrt{1+f'(x)^2}\,dx = \int_{-1/2}^{1/2} \sqrt{1+\frac{16\,x^2}{1-4\,x^2}}\,dx.$$

12. (10 points) Find the area of the surface obtained by rotating the curve $y = x^2$ for $1 \le x \le 2$ about the y-axis.

SOLUTION:

$$\begin{aligned} S &= \int_{x=1}^{x=2} 2\pi x\,ds \\ &= \int_1^2 2\pi x\sqrt{1+4x^2}\,dx \\ &= \frac{\left(17\sqrt{17} - 5\sqrt{5}\right)\pi}{6}. \end{aligned}$$