Practice Final Exam (a) with answers

Math 142

PART I

1. (28 points total) Evaluate the following integrals.

(a)

$$\int x^2\, e^{-x^3}\, dx$$

SOLUTION: Substitute $u = -x^3$ and get

$$\int x^2\, e^{-x^3}\, dx = \frac{-e^{-x^3}}{3}$$

(b)

$$\int_1^e \frac{\sqrt{\ln x}}{x}\, dx$$

SOLUTION: Substitute $u = \ln\, x$ and get

$$\begin{aligned}\int_1^e \frac{\sqrt{\ln x}}{x}\, dx &= \left.\frac{2\log(x)^{\frac{3}{2}}}{3}\right|_1^e \\ &= \frac{2}{3}\end{aligned}$$

(c)

$$\int_1^3 x\,(x^2+1)^{3/2}\, dx$$

SOLUTION: Substitute $u = -x^2 + 1$ and get

$$\begin{aligned}\int_1^3 x\,(x^2+1)^{3/2}\, dx &= \left.\frac{\sqrt{1+x^2}\,\left(\frac{2}{5} + \frac{4\,x^2}{5} + \frac{2\,x^4}{5}\right)}{2}\right|_1^1 \\ &= \frac{-4\,\sqrt{2}}{5} + 20\,\sqrt{10}.\end{aligned}$$

(d)

$$\int \frac{\sin t}{(\cos t+1)^2}\,dt$$

SOLUTION: Substitute $u = \cos\ t + 1$ and get

$$\int \frac{\sin t}{(\cos t+1)^2}\,dt \;=\; \frac{2\cos(\frac{t}{2})^2}{(1+\cos(t))^2}.$$

2. (8 points) Find the average value of $f(x) = x^3 + x$ on the interval $[1, 3]$.

SOLUTION:

$$\begin{aligned}\frac{1}{3-1}\int_1^3 x^3 + x\,dx &= \left.\frac{\frac{x^2}{2}+\frac{x^4}{4}}{2}\right|_1^3 \\ &= 12.\end{aligned}$$

3. (26 points) Let R be the region in the first quadrant bounded by the curves $y = x^2$ and $y = 4x - x^2$. Calculate:

(a) The area of R.

SOLUTION: The curves intersect at $(0, 0)$ and $(2, 4)$, so we must integrate from 0 to 2. We have

$$\begin{aligned}\int_0^2 (4x - x^2) - x^2\,dx &= \left. 2\,x^2 - \frac{2\,x^3}{3}\right|_0^2 \\ &= \frac{8}{3}.\end{aligned}$$

(b) The volume obtained by rotating R about the x-axis (slice method is recommended).

$$\begin{aligned}V &= \pi\int_0^2 (4x - x^2)^2 - (x^2)^4\,dx \\ &= \pi\left.\left(\frac{16\,x^3}{3} - 2\,x^4\right)\right|_0^2 \\ &= \frac{32\,\pi}{3}.\end{aligned}$$

(c) The volume obtained by rotating R about the y-axis (shell method is recommended).

$$\begin{aligned} V &= 2\pi \int_0^2 x\left((4x - x^2) - x^2\right)\,dx \\ &= \left.\frac{4\,x^3}{3} - \frac{x^4}{2}\right|_0^2 \\ &= \frac{8}{3}. \end{aligned}$$

4. (12 points) A rectangular swimming pool is 50 m long, 15 m wide, and 3 m deep. The depth of the water is 2 m. How much work is required to pump all of the water out over the side ? (Use $\rho = 1000$ kg/m^3 for the density of the water and $g = 10$ m/s^2 for the acceleration of gravity.)

SOLUTION:

$$W = \int_1^3 7500x\,dx = 3750x^2\Big|_1^3 = 30000\text{joules.}$$

5. (10 points) A particle moves along a line with velocity function $v(t) = t^2 - 2t - 8$. Find:

(a) The displacement during the time interval $[1, 6]$;

SOLUTION:

$$\begin{aligned} \int_1^6 t^2 - 2t - 8\,dt &= \left.-8\,t - t^2 + \frac{t^3}{3}\right|_1^6 \\ &= -\frac{10}{3}. \end{aligned}$$

(b) The distance traveled during the time interval $[1, 6]$.

SOLUTION: The velocity is negative for $t < 4$ and positive for $t > 4$, so the distance travelled is

$$\begin{aligned} \int_1^6 |t^2 - 2t - 8|\,dt &= -\int_1^4 t^2 - 2t - 8\,dt + \int_4^6 t^2 - 2t - 8\,dt \\ &= 18 + \frac{44}{3} = \frac{98}{3}. \end{aligned}$$

6. (16 points) If $f(x) = x^3 - 3x^2 + 5$,

(a) Find the intervals on which $f(x)$ is increasing.

SOLUTION: $(-\infty, 0)$ and $(2, \infty)$.

(b) Find the intervals on which $f(x)$ is decreasing. SOLUTION: $(0, 2)$.

(c) Find the local maxima of $f(x)$. SOLUTION: $x = 0$, $y = 5$.

(d) Find the local minima of $f(x)$. SOLUTION: $x = 2$, $y = 1$.

(e) Find the intervals on which $f(x)$ is concave up. SOLUTION: $(1, \infty)$.

(f) Find the intervals on which $f(x)$ is concave down . SOLUTION: $(-\infty, 1)$.

(g) Find the inflection points of $f(x)$. SOLUTION: $x = 1$, $y = 3$.

(h) Sketch the graph of $f(x)$.

PART II

7. (32 points) Evaluate the following integrals:

(a)

$$\int x^2 \sin 4x \, dx = \frac{-\left(\left(-1 + 8\,x^2\right)\,\cos(4\,x)\right)}{32} + \frac{x\,\sin(4\,x)}{8}$$

(b)

$$\int_1^e x^5 \ln x \, dx = \frac{1 + 5\,e^6}{36}$$

(c)

$$\int_3^4 \frac{1}{x^2 - 3x + 2} \, dx = 2\,\log(2) - \log(3)$$

(d)

$$\int \frac{1}{(x-1)(x^2+1)} \, dx = \frac{-\arctan(x)}{2} + \frac{\log(-1+x)}{2} - \frac{\log(1+x^2)}{4}$$

8. (16 points) Evaluate the following integrals:

(b) (use a trigonometric substitution)

$$\int \sqrt{9 + x^2} \, dx = \frac{\frac{2\,x\,\sqrt{9+x^2}}{9} - \log(-1 + \frac{x}{\sqrt{9+x^2}}) + \log(1 + \frac{x}{\sqrt{9+x^2}})}{4}$$

9. (16 points) Evaluate the following integrals or show that they are divergent:

(a)

$$\int_2^{10} \frac{1}{\sqrt[3]{x-2}} \, dx = 6$$

(b)

$$\int_0^{\infty} x\,e^{-2x} \, dx = \frac{1}{4}$$

10. (8 points) Set up Simpson's Rule sum with $n = 4$ subdivisions for the integral:

$$\int_0^4 \sqrt{1 + x^3}\, dx.$$

SOLUTION: Let $f(x) = \sqrt{1 + x^3}$. Then we have

$$\begin{aligned} \int_0^4 \sqrt{1 + x^3}\, dx &\approx \frac{1}{3}\left(f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right) \\ &= \frac{7 + 4\sqrt{2} + 8\sqrt{7} + \sqrt{65}}{3}. \end{aligned}$$

11. (12 points) Find the length of the curve $y = \frac{4}{3}x^{3/2},\ \ 0 \le x \le 2$.

SOLUTION: Let $f(x) = \frac{4}{3}x^{3/2}$. Then we have

$$\begin{aligned} s &= \int_0^2 \sqrt{1 + f'(x)^2}\, dx \\ &= \int_0^2 \sqrt{1 + 4x}\, dx \\ &= \left(\frac{1}{6} + \frac{2\,x}{3}\right)\sqrt{1 + 4\,x} \\ &= \frac{13}{3}. \end{aligned}$$

12. (16 points) Find the area of the surface obtained by rotating the curve $y = x^3$, $0 \le x \le 1$ about the x-axis.

SOLUTION: Let $f(x) = x^3$. Then we have

$$\begin{aligned} S &= \int_{x=0}^{x=1} 2\pi y\, ds \\ &= 2\pi \int_0^1 x^3\sqrt{1 + 9x^4}\, dx \\ &= 2\pi \int_0^1 x^3\,\sqrt{1 + 9\,x^4}\, dx \end{aligned}$$

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\begin{aligned}
&= \left. \frac{\pi \left(\frac{2}{27} + \frac{2\,x^4}{3}\right) \sqrt{1 + 9\,x^4}}{2} \right|_0^1 \\
&= 2\left(-\left(\frac{1}{54}\right) + \frac{5\sqrt{10}}{27}\right)\pi.
\end{aligned}
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