MATH 142

MIDTERM EXAM II ANSWERS

April 11, 2002

1. (10 pts) Find the area of the region enclosed by the curves $x = y^2 + 2$ and $x = 4$.

Setting $y^2 + 2 = 4$ yields $y = \pm\sqrt{2}$.

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} 4 - (y^2 + 2)\, dy = \int_{-\sqrt{2}}^{\sqrt{2}} 2 - y^2\, dy = 2y - \frac{1}{3}y^3 \Big|_{-\sqrt{2}}^{\sqrt{2}}$$

$$\text{ANSWER:} \quad 2\sqrt{2} - \frac{2}{3}\sqrt{2} - \left(-2\sqrt{2} + \frac{2}{3}\sqrt{2}\right) = \frac{8}{3}\sqrt{2}$$

2. (10 pts) A spring at rest has length of $2\,m$. Assuming that the spring constant k equals $10\,N/m$, calculate the work required to stretch the spring so as to increase its length to $4\,m$.

We use x to denote the amount by which the spring is stretched beyond its rest length.

$$W = \int_0^2 10x\, dx = 5x^2 \Big|_0^2 = 20$$

Here we have made use of the Hooke's Law which says that $F(x) = kx$.

$$\text{ANSWER:} \quad 20\,Nm$$

3. (20 pts) Find the volumes of the solids obtained by rotating the specified regions about the given axes.

(a) enclosed by $y = \sqrt{x-1}$, $x = 1$, and $y = 1$; about $y = 0$

This can be done using slices. Setting $1 = \sqrt{x-1}$ yields $x = 2$.

$$V = \int_1^2 \pi \left(1^2 - \left(\sqrt{x-1}\,\right)^2\right)\, dx = \pi \int_1^2 2 - x\, dx = \pi \left(2x - \frac{1}{2}x^2\right) \Big|_1^2$$

$$\text{ANSWER:} \quad \pi\left(4 - 2 - 2 + \frac{1}{2}\right) = \frac{\pi}{2}$$

(b) enclosed by $y = x^2$ and $y = 3x$; about $x = 4$

This can be done using shells. Setting $x^2 = 3x$ yields $x = 0, 3$.

$$V = \int_0^3 2\pi(4-x)(3x-x^2)\,dx = 2\pi \int_0^3 12x - 7x^2 + x^3\,dx = 2\pi\left(6x^2 - \frac{7}{3}x^3 + \frac{1}{4}x^4\right)\Big|_0^3$$

ANSWER: $2\pi\left(54 - 63 + \frac{1}{4}81\right) = \frac{45\pi}{2}$

4. (10 pts) A 100 foot tower has cross sectional areas which, at height y, are given by $A(y) = \dfrac{y}{1+y^2}$. Find its volume.

$$V = \int_0^{100} A(y)\,dy = \int_0^{100} \frac{y}{1+y^2}\,dy = \frac{1}{2}\ln\left|1+y^2\right|\Big|_0^{100}$$

ANSWER: $\dfrac{\ln 10001}{2}$

5. (10 pts) Find the average value of $f(x) = x^2\sqrt{1+x^3}$ over $[0, 2]$.

$$f_{ave} = \frac{1}{2-0}\int_0^2 x^2(1+x^3)^{1/2}\,dx = \frac{1}{2}\int_1^9 u^{1/2}\,\frac{du}{3} = \frac{1}{6}\,\frac{2}{3}u^{3/2}\Big|_1^9$$

Here we performed substitution $u = 1 + x^3$ with which $x^2\,dx = \dfrac{du}{3}$.

ANSWER: $\frac{1}{9}(9^{3/2} - 1) = \frac{26}{9}$

6. (24 pts) Evaluate the following indefinite integrals.

(a) $\displaystyle\int 2x\cos x^2\,dx$

We use substitution $u = x^2$ with which $du = 2x\,dx$.

$$\int 2x\cos x^2\,dx = \int \cos u\,du = \sin u + C$$

ANSWER: $\sin x^2 + C$

(b) $\int x^2 \cos x \, dx$

We use integration by parts twice. First, let $u = x^2$ and $dv = \cos x \, dx$ so that $du = 2x \, dx$ and $v = \sin x$.

$$\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx$$

Second, let $u = 2x$ and $dv = \sin x \, dx$ so that $du = 2 \, dx$ and $v = -\cos x$.

$$x^2 \sin x - \int 2x \sin x \, dx = x^2 \sin x - \left(-2x \cos x - \int -2 \cos x \, dx \right)$$

ANSWER: $x^2 \sin x + 2x \cos x - 2 \sin x + C$

(c) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$

We use substitution $u = \sqrt{x}$ with which $du = \frac{1}{2\sqrt{x}} \, dx$.

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx = \int e^u \, 2du = 2e^u + C$$

ANSWER: $2e^{\sqrt{x}} + C$

(d) $\int \arcsin x \, dx$

We integrate by parts. Let $u = \arcsin x$ and $dv = dx$ so that $du = \frac{1}{\sqrt{1-x^2}} \, dx$ and $v = x$.

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx$$

Next, we perform substitution $s = 1 - x^2$ with which $ds = -2x \, dx$.

$$\begin{aligned}
\int \arcsin x \, dx &= x \arcsin x - \int \frac{1}{s^{1/2}} \left(-\frac{ds}{2} \right) \\
&= x \arcsin x + \frac{1}{2} \int s^{-1/2} \, ds \\
&= x \arcsin x + \frac{1}{2} \frac{2}{1} s^{1/2} + C
\end{aligned}$$

ANSWER: $x \arcsin x + \sqrt{1-x^2} + C$

7. (16 pts) Evaluate the following indefinite integrals.

(a) $\displaystyle\int \sin^3 x \cos^5 x \, dx$

$$\begin{aligned}
\int \sin^3 x \cos^5 x \, dx &= \int \sin^3 x \cos^4 x \cos x \, dx \\
&= \int \sin^3 x (1 - \sin^2 x)^2 \cos x \, dx \quad (u = \sin x;\ du = \cos x \, dx) \\
&= \int u^3 (1 - u^2)^2 \, du \\
&= \int u^3 (1 - 2u^2 + u^4) \, du \\
&= \int u^3 - 2u^5 + u^7 \, du \\
&= \frac{1}{4} u^4 - \frac{1}{3} u^6 + \frac{1}{8} u^8 + C
\end{aligned}$$

ANSWER: $\displaystyle \frac{1}{4} \sin^4 x - \frac{1}{3} \sin^6 x + \frac{1}{8} \sin^8 x + C$

(b) $\displaystyle\int \tan^3 x \sec^4 x \, dx$

$$\begin{aligned}
\int \tan^3 x \sec^4 x \, dx &= \int \tan^3 x \sec^2 x \sec^2 x \, dx \\
&= \int \tan^3 x (1 + \tan^2 x) \sec^2 x \, dx \quad (u = \tan x;\ du = \sec^2 x \, dx) \\
&= \int u^3 (1 + u^2) \, du \\
&= \int u^3 + u^5 \, du \\
&= \frac{1}{4} u^4 + \frac{1}{6} u^6 + C
\end{aligned}$$

ANSWER: $\displaystyle \frac{1}{4} \tan^4 x + \frac{1}{6} \tan^6 x + C$