# Project III - Determinants

1. Let $A_n$ be the $n \times n$ matrix
$$\begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & 0 & \\ & 1 & 2 & & & \\ & & & \cdots & & \\ & 0 & & & 2 & 1 \\ & & & & 1 & 2 \end{bmatrix}$$
   (a) Compute $\det A_n$ for $n = 1,\ 2,\ 3$, and 4.
   (b) Make a conjecture about $a_n = \det A_n$ for $n \in \mathbb{N}$.
   (c) Find a recursive equation on $a_n$'s by expanding $A_n$ down the first row, and then expanding one of the obtained matrices across the first row.
   (d) Use the recursive equation from part (c) to prove your conjecture about $\det A_n$ by induction.

2. Let $B_n$ be the $n \times n$ matrix
$$\begin{bmatrix} 5 & 2 & & & & \\ 2 & 5 & 2 & & 0 & \\ & 2 & 5 & & & \\ & & & \cdots & & \\ & 0 & & & 5 & 2 \\ & & & & 2 & 5 \end{bmatrix}$$
   (a) Compute $b_n = \det B_n$ for $n = 1,\ 2,\ 3$, and 4. Notice that there is no obvious pattern.
   (b) Find a recursive equation on $b_n$'s.
   (c) Show that if $\alpha$ is a root of the equation $x^2 - c_1 x - c_2 = 0$, then the sequence $\{b_n | b_n = \alpha^n\}$ satisfies the recursive equation $b_n = c_1 b_{n-1} + c_2 b_{n-2}$ (1)
   (d) Show that the space of all sequences $\{b_n\}$ satisfying (1) has dimension 2.
   (e) Show that if $\alpha \neq \beta$, then the sequences $\{\alpha^n\}$ and $\{\beta^n\}$ are linearly independent.
   (f) Conclude that if the equation $x^2 - c_1 x - c_2 = 0$ has two distinct roots, $\alpha$ and $\beta$, then any sequence satisfying (1) is a linear combination of the sequences $\{\alpha^n\}$ and $\{\beta^n\}$.
   (g) Solve the quadratic equation associated with the recursive equation you obtained in part (c), and find a linear combination of the sequences $\{\alpha^n\}$ and $\{\beta^n\}$ that gives $\{b_n\}$.

3. Convince yourself that the method of problem 2 will work for any $n \times n$ matrix
$$C_n = \begin{bmatrix} a & b & & & & \\ b & a & b & & 0 & \\ & b & a & & & \\ & & & \cdots & & \\ & 0 & & & a & b \\ & & & & b & a \end{bmatrix}$$
   such that the quadratic equation assiciated with the recursive equation on $\det C_n$ has two distinct roots, even if they are irrational or complex.

4. Now consider the situation when the quadratic equation associated with the recursive equation on $\det C_n$ has one root (of multiplicity 2).
   (a) Show that this occurs if and only if $a = \pm 2b$.
   (b) Show that in this case $\det C_n = b^n(n+1)$.