**Math 235** **Summer 2001**

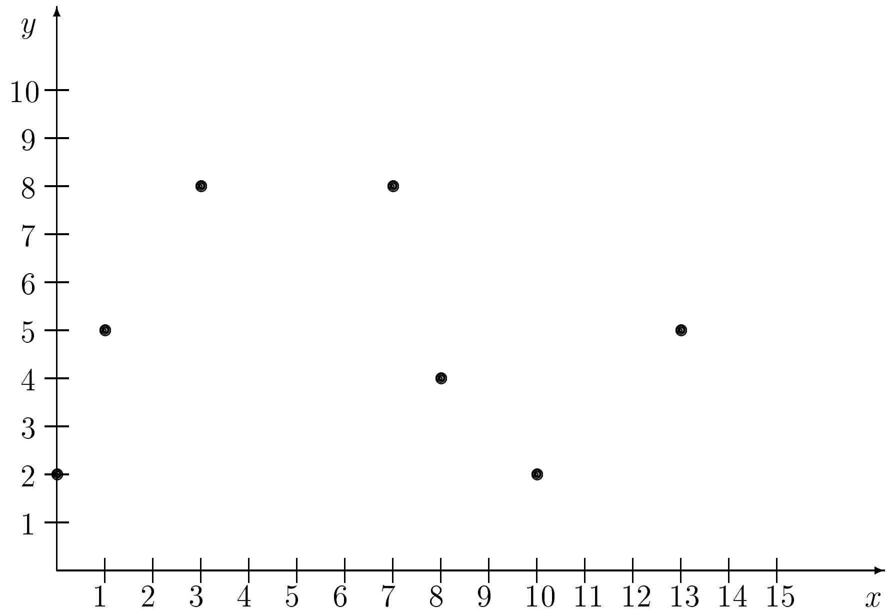
# Project IV, Interpolations

Note: In this project, you will find some hints on how you can use Mathematica. But you are welcome to use any other computer softwear if you wish.

1. Do problem 35 on page 26 of your book.

   When computers are used to generate images, there is often the need to "interpolate" between grid points to provide a smooth looking curve which fits the points given. One solution to this problem it to use "Cubic Splice Interpolation".

   Consider the collection of points in $\mathbb{R}^2$: { (0,2), (1,5), (3,8), (7,8), (8,4), (10,2), (13,5) }; there is a plot of these below. We wish to create a curve going through these points, and one which looks smooth and not too curvey.




   Definition. A cubic is a polynomial of the form $ax^3 + bx^2 + cx + d$.

2. Show that given any four points with different $x$-coordinates there is one and only one cubic going through these four points.
   Hint: Show that

$$\det \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{bmatrix} = \prod_{1 \le i < j \le 4} (x_j - x_i).$$

   It follows that if $x_1$, $x_2$, $x_3$, and $x_4$ are different numbers, then the determinant of the above matrix is non-zero, thus the linear system on $a$, $b$, $c$, and $d$ has a unique solution.

3. (a) Use linear algebra to find the coefficients $a$, $b$, $c$, and $d$ for the cubic $p_1(x)$ which fits the first four points given (you should have four equations with four unknowns).

   (b) Find the cubic $p_2(x)$ fitting the 4th through 7th points.

   (c) Draw a graph of the cubics $p_1(x)$ and $p_2(x)$. You can use Mathematica to do this:

```
Plot[f, {x, xmin, xmax}, PlotRange -> {ymin, ymax}]
```

Fo example, if you had found $p_1(x) = \frac{1}{2}x^3 - 2x^2 + \frac{3}{7}x + 2$ then looking at the points it goes through you should use

$$\texttt{g1} = \texttt{Plot}[(1/2)\texttt{x}^3 - 2\texttt{x}^2 + (3/7)\texttt{x} + 2,\ \{\texttt{x}, -1, 7\}, \texttt{Plotrange}->\{-2, 9\}]$$

This will assign g1 to the plot. Then assign g2 to the plot of the second polynomial with $x$ range being 7 to 13, and the $y$ range being the same as before. Now to put the two plots on the same axes, use

$$\texttt{Show}[\texttt{g1}, \texttt{g2}]$$

You will notice that the cubics look nice, but the point where they join together is unsatisfactory - there is a "kink" at the join. The problem is that the cubics agree, but their first derivatives do not. To overcome this we shall keep the firest cubic we found, and then find a second cubic which goes through the 4th through 6th points - thats' three conditions for four unknowns. For the fourth condition we shall require that the first derivative of the two cubics agree at the 4th point.

4. Find $q_2(x)$, our improvement on $p_2(x)$, which goes through points 4, 5, and 6, and which has $p_1'(7) = q_2'(7)$ at fourth point. Again use linear algebra to find the coefficients of the cubic. Assign h2 to the plot of this improved second polynomial - use 7 to 10 for the x range, and be sure to use the same $y$ range as before. Use `Show[g1,h2]` to plot the two polynomials $p_1(x)$ and $q_2(x)$.

The kink is a little less severe, but not great, and the second cubic gets very curvey in its effort to match the conditions we have set. There is a better solution, the standard Cubic Spline: this Spline matches only two points at a time, together with the first and second derivatives of touching cubics mathching at the point of contact. We shall find the spline through the first 6 points; given that we have 6 points, we will need 5 cubics (why?). That means there are $5 \times 4 = 20$ undetermined coefficients $a_1, \ldots, a_{20}$ say, where $a_1$, $a_2$, $a_3$, $a_4$ are the coefficients of the first cubic, $a_5$, $a_6$, $a_7$, $a_8$ of the second, and so on.

Forcing each cubic to go through two specified points results in $5 \times 3 = 10$ conditions. Forcing the first derivatives to match at each interior point (i.e. not first not last) results in 4 more conditions, and then the second derivatives matching gives 4 more. That gives $10+4+4 = 18$ conditions. The last two we specify are that the second derivative of the first point iszero, and that the second derivative of the last cubic at the last point is zero. This gives us the 20 conditions.

What does the linear system look like? The $20 \times 20$ matrix $M$ is shown below - the first 10 rows match the cubics to the points, the next 4 match the first derivatives, the next four match the second derivatives, and the last two set the second derivative at $x_1$ and $x_8$ to be zero. All the "missing" entries are zero.

$$
\left[\begin{array}{cccccccccccccccccccc}
x_1^3 & x_1^2 & x_1 & 1 \\
x_2^3 & x_2^2 & x_2 & 1 \\
& & & & x_2^3 & x_2^2 & x_2 & 1 \\
& & & & x_3^3 & x_3^2 & x_3 & 1 \\
& & & & & & & & x_3^3 & x_3^2 & x_3 & 1 \\
& & & & & & & & x_4^3 & x_4^2 & x_4 & 1 \\
& & & & & & & & & & & & x_4^3 & x_4^2 & x_4 & 1 \\
& & & & & & & & & & & & x_5^3 & x_5^2 & x_5 & 1 \\
& & & & & & & & & & & & & & & & x_5^3 & x_5^2 & x_5 & 1 \\
& & & & & & & & & & & & & & & & x_6^3 & x_6^2 & x_6 & 1 \\
3x_2^2 & 2x_2 & 1 & 0 & -3x_2^2 & -2x_2 & -1 & 0 \\
& & & & 3x_3^2 & 2x_3 & 1 & 0 & -3x_3^2 & -2x_3 & -1 & 0 \\
& & & & & & & & 3x_4^2 & 2x_4 & 1 & 0 & -3x_4^2 & -2x_4 & -1 & 0 \\
& & & & & & & & & & & & 3x_5^2 & 2x_5 & 1 & 0 & -3x_5^2 & -2x_5 & -1 & 0 \\
6x_2 & 2 & 0 & 0 & -6x_2 & -2 & 0 & 0 \\
& & & & 6x_3 & 2 & 0 & 0 & -6x_3 & -2 & 0 & 0 \\
& & & & & & & & 6x_4 & 2 & 0 & 0 & -6x_4 & -2 & 0 & 0 \\
& & & & & & & & & & & & 6x_5 & 2 & 0 & 0 & -6x_5 & -2 & 0 & 0 \\
6x_1 & 2 & 0 & 0 \\
& & & & & & & & & & & & & & & & 6x_6 & 2 & 0 & 0
\end{array}\right]
$$

5. (a) Make sure you can explain every entry of the above matrix.

   (b) Write out the vector $b$ of the system $Ma = b$ that you have to solve in order to find the coefficients of the cubics.

   (c) Use a computer to solve the system.

   (d) To define the 5 plots for these cubics, you can use a loop as follows. First define a vector of the $x$-values of the points our spline goes through:

   $$\texttt{xpts} = \{0, 1, 3, 7, 8, 10\}$$

   Now define, for $i = 1, 2, \ldots, 5$, `sp[i]` to be the $i^{th}$ plot via the syntax

   $$\texttt{Do[expression, \{i, imax\}]}$$

   The $i^{th}$ cubic is made up of the components $4i-3$, $4i-2$, $4i-1$, and $4i$ of the vector $a$ (why?). The syntax for specifying the $n^{th}$ element in $a$ is `a[[n]]`. The loop becomes:

   $$\texttt{Do[sp[i] = Plot[a[[4i - 3]]x}^3\texttt{+a[[4i - 2]]x}^2\texttt{+a[[4i - 1]]x + a[[4i]],}$$
   $$\texttt{\{x, xpts[[i]], xpts[[i + 1]]\}, Plotrange- > \{0, 15\}], \{i, 5\}]}$$

   The $\texttt{\{x, xpts[[i]], xpts[[i + 1]]\}}$ specifies that the $i^{th}$ plot runs from $x_i$ to $x_{i+1}$, the $\texttt{PlotRange- > \{0, 15\}}$ specifies the range of $y$-values to plot, and the $\texttt{\{i, 5\}}$ specifies that $i$ should run from 1 to 5.

   (e) Plot the 5 cubics together:

   $$\texttt{Show[sp[1], sp[2], sp[3], sp[4], sp[5]]}$$