Math 235 **Summer 2001**

Project VIII, Linear Transformations

Note: Giving answers "yes" or "no" is not sufficient. Provide proofs for all your statements.

Let $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, and $T(M) : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by M, that is, $T(M)$ is the multiplication by M on the left.

1. (a) Is $T(M) : \mathbb{R}^2 \to \mathbb{R}^2$ one-to-one?
   (b) Is $T(M) : \mathbb{R}^2 \to \mathbb{R}^2$ onto?
   (c) What is the necessary and sufficient condition on $M \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ for $T(M) : \mathbb{R}^n \to \mathbb{R}^n$ to be one-to one and onto?

Let $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ again. Notice that if $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ is such that x and y are integers, that is, if $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{Z}^2$, then $M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 3y + 4y \end{bmatrix} \in \mathbb{Z}^2$. Thus we can restrict $T(M)$ onto $\mathbb{Z}^2$, and consider the map $T(M) : \mathbb{Z}^2 \to \mathbb{Z}^2$.

2. (a) Is $T(M) : \mathbb{Z}^2 \to \mathbb{Z}^2$ one-to one?
   (b) Is $T(M) : \mathbb{Z}^2 \to \mathbb{Z}^2$ onto?
   (c) If the answer for (a) is "yes", give an example of a $2\times 2-$matrix M' s.t. $T(M') : \mathbb{Z}^2 \to \mathbb{Z}^2$ is not one-to-one.
   If the answer for (a) is "no", give an example of M' s.t. $T(M') : \mathbb{Z}^2 \to \mathbb{Z}^2$ is one-to-one.
   (d) If the answer for (b) is "yes", give an example of a $2\times 2-$matrix M'' s.t. $T(M'') : \mathbb{Z}^2 \to \mathbb{Z}^2$ is not onto.
   If the answer for (b) is "no", give an example of M'' s.t. $T(M'') : \mathbb{Z}^2 \to \mathbb{Z}^2$ is onto.

Let $a,\ b,\ c$, and d be integers, and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

3. (a) Find a necessary and sufficient condition on $a,\ b,\ c$, and d for $T(M) : \mathbb{Z}^2 \to \mathbb{Z}^2$ to be one-to-one.
   (b) Find a necessary and sufficient condition on $a,\ b,\ c$, and d for $T(M) : \mathbb{Z}^2 \to \mathbb{Z}^2$ to be onto.
4. (a) Generalize your results in problem 3 onto the case of $n \times n$ matrices with integer coefficients.
   (b) Compare your answers in 4(a) and 1(c).