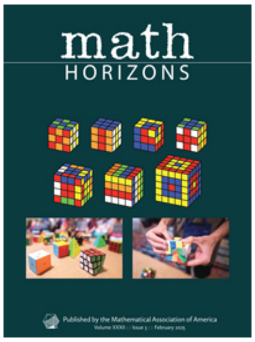

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Permutations in Rubik's Cubes

Maria Nogin, Katherine Nogin, and Michelle Nogin

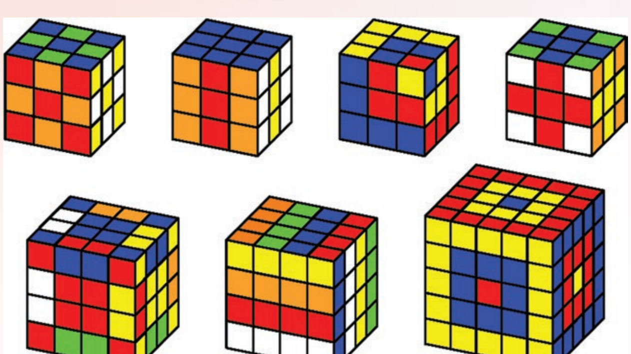
Last year, we celebrated 50 years since the creation of Rubik's cube.

Perhaps most people have tried to solve a Rubik's cube at least once in their life. Some dedicate a lot of time to learning efficient methods. Speed cubers know many different algorithms and can do them really fast: The world record when this article was written was about three seconds. (This time is beyond us!) Most solvers, though, know just a handful of algorithms and take a few minutes. (Or longer! Why rush if we enjoy the process, right?)

However, there are other things we can do with a Rubik's cube besides solving it. One of them is creating various patterns, such as those shown in figure 1. The bigger the cube, the more elaborate patterns one can create.

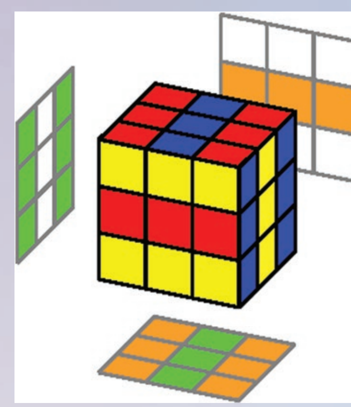
But can we create any pattern we like? Observe that a solved cube has six faces, all colored differently. For a cube with the standard coloring, white and yellow are on opposite faces, red and orange are on opposite faces, and blue and green are as well. Thus, there does not exist an edge piece with, say, white on one side and yellow on the other. Likewise, there is no corner piece with red, orange, and blue on its three sides. But what about those configurations that only involve pieces that actually exist, and each only once? A closer inspection of any odd-sized cube reveals that there is no way to swap face centers as they are connected in a rigid manner. When the front center is red and the top center is blue, the one on the right must be yellow, the one on the left must be white, and so on. Let's say this

Figure 1. A few patterns in Rubik's cubes.



condition is satisfied as well. Now can we create any pattern? For example, can we create the pattern shown in figure 2? It is easy to verify that all corner and edge pieces exist, but can they be moved to make this configuration?

Figure 2. Is it possible to make this stripe pattern?



As we will see, it turns out that not every configuration is attainable. For example, in any odd-sized cube, it is not possible to switch just two small pieces of the cube while leaving all other pieces intact. Why is that?

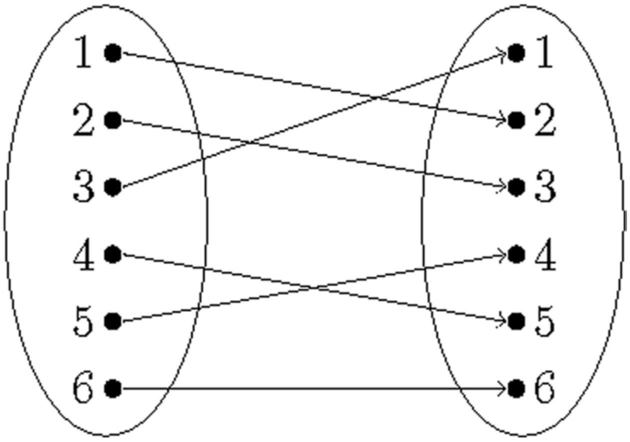
To investigate this question, we need to use some concepts and facts that one might see in an abstract algebra course.

Permutations

A *permutation* on a set $S = \{1,2,\ldots,n\}$ is, essentially, some rearrangement of all the elements of S. More formally, a permutation is a one-to-one and onto function from S to itself. Permutations can be described by using cycle notation, where elements that are cyclically permuted are written in a list enclosed with parentheses; no element appears more than once. If an element is sent to itself, it can be omitted from the notation. For example, the permutation pictured in figure 3 sends 1 to 2, 2 to 3, 3 to 1, 4 to 5, 5 to 4, and 6 to itself, so it is denoted (1,2,3)(4,5). If a permutation cyclically permutes k elements and sends the rest of the elements to themselves, it is called a *cycle* of length k. For example, (1,2,3,4) is a cycle of length 4. Note that because permutations are functions, they can be composed. We compute the composition of permutations reading them from right to left, just like we would compose functions in general. For example, if $\sigma = (1,2)$ and $\tau = (2,3)$, then

$$(\sigma \circ \tau)(1) = \sigma(\tau(1)) = \sigma(1) = 2,$$
$$(\sigma \circ \tau)(2) = \sigma(\tau(2)) = \sigma(3) = 3, \text{ and}$$
$$(\sigma \circ \tau)(3) = \sigma(\tau(3)) = \sigma(2) = 1;$$

Figure 3. Permutation (1,2,3)(4,5).



thus, $\sigma \circ \tau = (1,2,3)$. When writing compositions of permutations in the cycle notation, we will omit the symbol $\circ$ and just write $(1,2)(2,3) = (1,2,3)$.

A cycle of length 2 is called a *transposition*. With a little thought, one can see that any permutation can be written as a composition of transpositions. Moreover, there are many ways to write a permutation as a composition of transpositions. For example, we invite you to verify that the composition $(1,2)(1,3)(2,3)(1,2)$ gives the same permutation as the composition $(1,2)(2,3)$, namely, the cycle $(1,2,3)$. However, as a theorem in abstract algebra states, any permutation can either only be written as a composition of an even number of transpositions or only as a composition of an odd number of transpositions. A permutation is called *even* (respectively, *odd*) if it can be written as a composition of an even (respectively, odd) number of transpositions. Being even or odd is referred to as the *parity* of the permutation.

Now let's see what happens to the parity when we compose two permutations. Suppose one permutation can be written as a composition of k transpositions and another permutation can be written as a composition of l transpositions. The parities of these two permutations correspond to those of k and l, respectively. When we compose these two permutations, we obtain the composition of all the $k+l$. transpositions. Thus, the parity of their composition is that of $k+l$. As the sum of two even numbers or two odd numbers is even and the sum of an even number and an odd number is odd, we see that the composition of two even permutations or two odd permutations is even, whereas the composition of an even permutation and an odd permutation is odd. This is an important fact to keep in mind when we get back to investigating cubes!

Practice your newfound permutation skills by verifying that

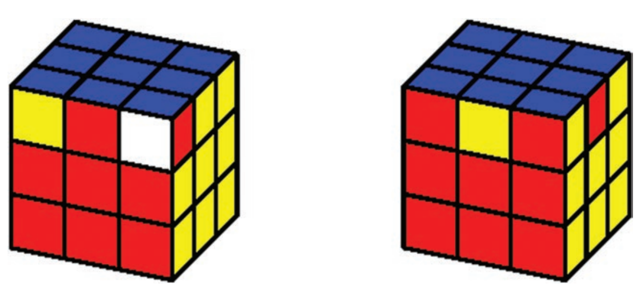
$$(a_1, a_2, \ldots, a_n) = (a_1, a_2)(a_2, a_3)\ldots(a_{n-1}, a_n).$$

This shows that a cycle of length n can be written as a composition of $n-1$ transpositions. This tells us that a cycle of an odd length is an even permutation and a cycle of an even length is an odd permutation. You might wish to read that last sentence one more time.

Switching Pieces

Let's take a closer look at the classical $3 \times 3 \times 3$ Rubik's cube. As you begin playing with the cube, all your moves will involve rotating faces. As you rotate a face, the center piece remains in the center. Now, hold the cube so the face centers do not move and notice how the other eight pieces (four corners and four edge pieces) move when a face is rotated. A 90° rotation cyclically permutes the four corners and cyclically permutes the four edge pieces. Each cycle of length 4 is an odd permutation, so the rotation, which is the composition of these two cycles, is an even permutation. As each single move of the cube is an even permutation, any number of moves can only create an even permutation! So, if we start with a solved cube, assuming it was solved when we first got it out of the package, we will always have an even permutation of the 20 movable pieces (eight corners and 12 edge pieces). Therefore, it is impossible to achieve a configuration in which the whole cube is solved but just two pieces are switched—whether two corners or two edge pieces—as shown in figure 4.

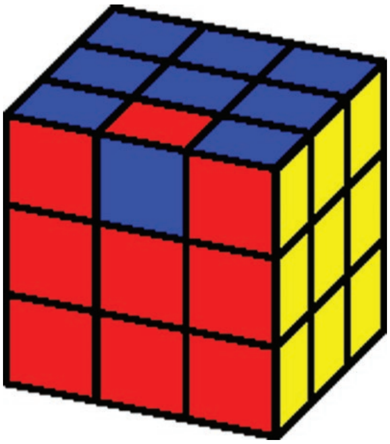
Figure 4. It is impossible to switch just two corners or just two edge pieces and have the rest of the cube solved.



Rotating Pieces

What about rotating pieces without moving them? Could we rotate any number of pieces any way we want? To understand the rotations of the edge pieces, we will consider a different kind of

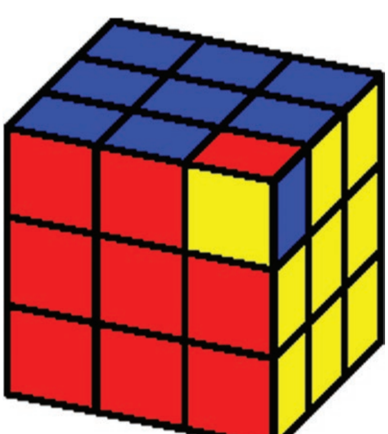
Figure 5. Is it possible to rotate a single edge piece?



permutation: that of the individual faces of the movable pieces rather than that of the whole pieces. For example, if the front-top edge piece is rotated, the two faces of that piece switch places, yielding a single transposition on the faces. Each move, a rotation of one face of the big cube by 90°, permutes a total of 20 small faces of the pieces. More precisely, it contains five cycles of length 4, so altogether it is an odd permutation. Two such rotations would be an even permutation. Thus, as we play with the cube, both odd and even permutations of these smaller faces are possible. Does this mean it is possible to rotate a single edge piece as shown in figure 5? It might seem so at first glance but be careful. The ability to produce *some* odd permutation does not guarantee that we can produce *any* odd permutation.

Let's restrict our attention to the permutation of the faces of just the edge pieces. There are 12 edges, so altogether they have 24 faces. Any 90° rotation of a face of the whole cube cyclically permutes four edge faces on the rotated face of the whole cube and cyclically permutes four edge faces on the outer perimeter of the rotated face. Thus, the move is an even permutation of the edge faces. It follows that a configuration where one edge piece is rotated incorrectly, and all other edge pieces are correctly situated, is impossible.

Figure 6. Is it possible to rotate a single corner?



Similarly, we could ask whether it is possible to rotate a single corner as shown in figure 6. We will leave this question to an interested reader.

Striped Pattern

Now that we understand some impossible configurations, we are well equipped to take another look at figure 2. Is that configuration possible? (Before reading further, feel free to investigate this pattern on your own.) Let's always hold our cube so that the red and blue centers are at the front and top, respectively, and track where we want the corner and edge pieces to go. Figure 7 has some of these pieces labeled in both the solved cube and the desired configuration.

Using B, G, O, R, W, and Y to denote the colors blue, green, orange, red, white, and yellow, we see that six corners move in two cycles of length 3—(BOY, GRY, BRW) and (BOW, GOY, GRW)—and the corners BRY and GOW remain in their original positions. The edge pieces move in two cycles of length 3—(BR, RY, BY) and (OW, GW, GO)—and one cycle of length 6—(BO, OY, GY, GR, RW, BW). Because a cycle of length 3 is an even permutation and a cycle of length 6 is an odd permutation, the composition of these is an odd permutation. That is impossible! So this visually appealing configuration is, unfortunately, unattainable.

Other Cubes

All of these results are valid on any odd-sized cube: $5\times5\times5$, $7\times7\times7$, and so on. The even-sized cubes are very different, though, because they do not have a single piece at the center of any edge. But you could pretend that they are there, only you can't see them! Therefore, if you switch or rotate some of them, you won't notice. So, what happens now? As before, if our previous argument does not apply, it does not yet mean that it *is* possible to switch two corners, for example, as shown in figure 8. However, it turns out that this is possible in any even-sized cube.

Figure 7. Comparing the solved cube to the stripe pattern in figure 2.

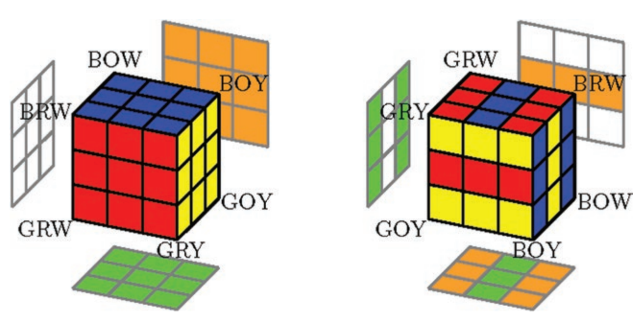
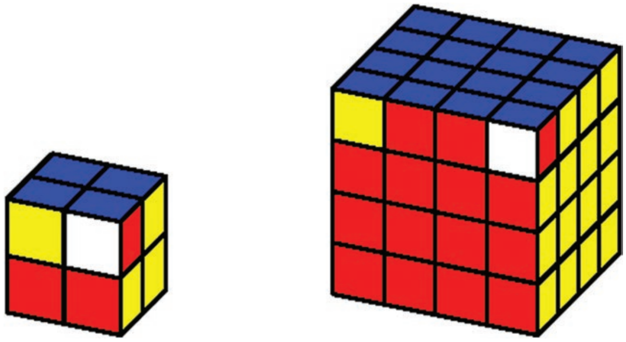


Figure 8. It is possible to switch just two corners in any even-sized cube.

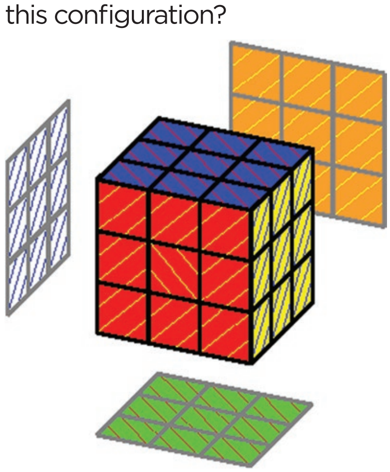


In addition, in any cube larger than $3 \times 3 \times 3$, a new kind of move is possible: rotating one of the middle layers or slices. It leaves all the corners intact but is an odd permutation on the edge pieces. It also permutes some of the interior face pieces, but many of those are indistinguishable! Not surprisingly, it gets more complicated with increasing the size of the cube, and that is why solving a bigger cube is not just a matter of time but requires knowing more algorithms. The good news is that if you learn to solve $4 \times 4 \times 4$ and $5 \times 5 \times 5$ cubes, then you can quite easily generalize those algorithms to the larger cubes, so after that, it does become just a matter of time.

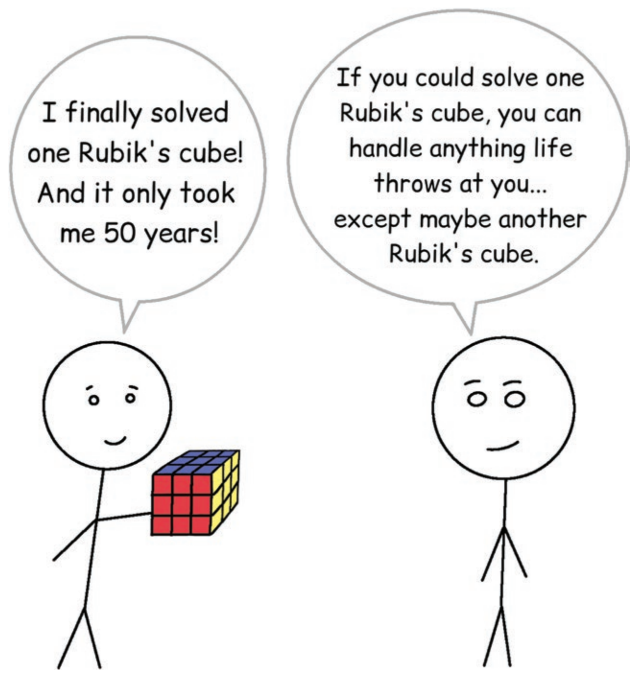
Here are a few questions for the reader to think about. Suppose we draw stripes (or some other pattern) on all faces of a solved $3 \times 3 \times 3$ cube. Is it possible to obtain a configuration in which the cube is solved but the stripes do not align on exactly one face as shown in figure 9?

Do the preceding arguments generalize to rectangular prisms that are not cubes, for example, $1 \times 2 \times 3$, $2 \times 2 \times 3$, or $2 \times 3 \times 3$? Can you use your knowledge of permutations to analyze a pyraminx and megaminx (tetrahedron and dodecahedron, respectively; you can find their pictures on the Web if you do not know what these look like)? Can you argue that certain permutations of the corners or edge pieces are impossible in these puzzles?

Figure 9. Is it possible to obtain this configuration?



There are many other uses of permutations besides Rubik's cubes. They appear in almost any branch of mathematics, from combinatorics to geometry, as well as other sciences. For example, in computer science, they are used to generate coding and decoding algorithms; in physics, to describe states of particles; and in chemistry, to study the geometric structure of molecules. Thus, Rubik's cubes are more than just fun puzzles; they are a gateway to higher mathematics applicable to many fields. ●



***Maria Nogin** is a professor of mathematics at Fresno State. She loves problem solving and discovering patterns in and connections between academic and recreational math. She is the founder of the Fresno Math Circle, an enrichment program for K–12 students, where she enjoys teaching mathematics through problems, games, and puzzles.*

***Katherine** and **Michelle** are Maria's daughters, currently attending Northwestern University and Clovis North High School, respectively. Both of them share Maria's passion for Rubik's cubes and mathematics in general.*

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