It is not a coincidence! On patterns in some Calculus optimization problems.

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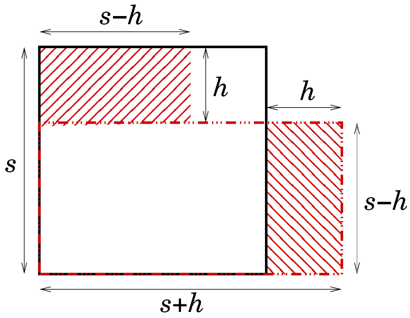
Outline

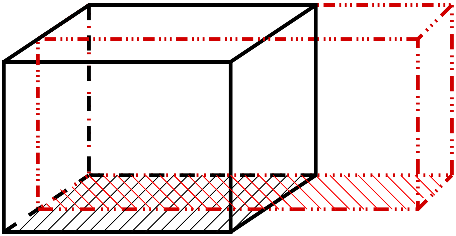
Optimizing rectangle

Out of all rectangles with a given perimeter, which one has the greatest area?



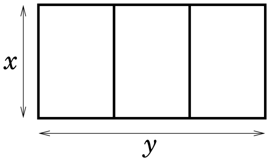
Optimizing rectangular box

Out of all rectangular boxes with a given volume, which one has the smallest surface area?



The rectangular field roblem

A farmer wants to fence off a rectangular field and divide it into 3 pens with fence parallel to one pair of sides. He has a total 2400 ft of fencing. What are the dimensions of the field has the largest possible area?



$y = \frac{2400-4x}{2} = 1200 - 2x$

$\text{Area}(x) = 1200x - 2x^2$

$\text{Area}'(x) = 1200 - 4x^2 = 0$

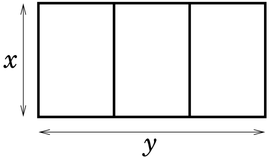
$x = 300$ is an absolute maximum

$y = 600$

Observation: the total length of vertical pieces: 1200 ft
the total length of horizontal pieces: 1200 ft

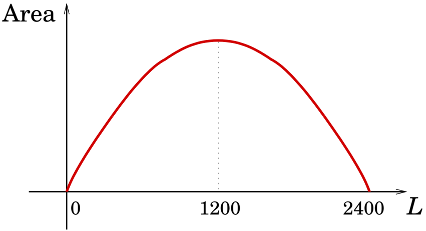
These are equal!

Why? Functional explanation

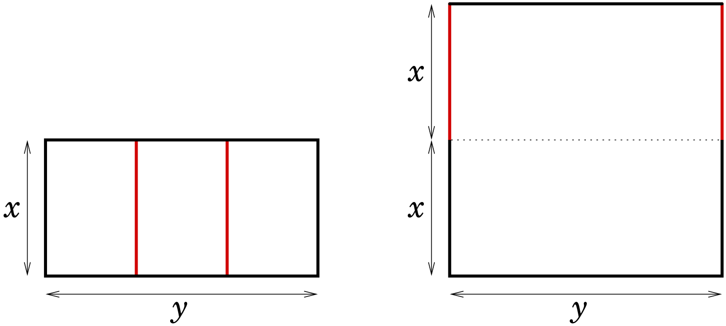


Let L be the total length of the vertical pieces.
$2400 - L$ is the total length of the horizontal pieces.

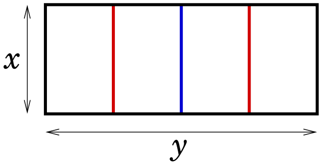
$$x = \frac{L}{4}, \quad y = \frac{2400-L}{2}, \quad \text{Area}(L) = \frac{L}{4} \cdot \frac{2400-L}{2}$$

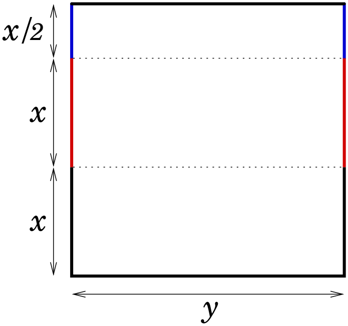


Why? Geometric explanation



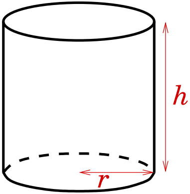
Why? Geometric explanation





The can problem

A cylindrical can has to have volume 1000cm^3. Find the dimensions of the can that minimize the amount of material used (i.e. minimize the surface area).



$$h = \frac{1000}{\pi r^2}$$

$$SA(r) = 2\pi r^2 + \frac{2000}{r}$$

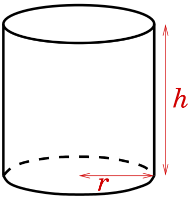
$$SA'(r) = 4\pi r - \frac{2000}{r^2} = 0$$

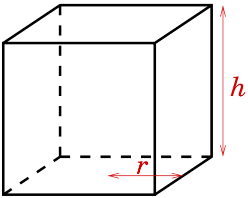
$$r = \sqrt[3]{\frac{500}{\pi}} \text{ is an absolute minimum}$$

$$h = 2\sqrt[3]{\frac{500}{\pi}}$$

Observation: $d = 2\sqrt[3]{\frac{500}{\pi}}$ cm $\quad$ $d = h!$

Why?





$$V_{can} = A_{circle} h$$

$$V_{cube} = A_{square} h$$

$$V_{cube} = \frac{A_{square}}{A_{circle}} V_{can} = \frac{4}{\pi} V_{can}$$

$$SA_{can} = 2A_{circle} + P_{circle} h$$

$$SA_{cube} = 2A_{square} + P_{square} h$$

Question: is $\frac{P_{square}}{P_{circle}} = \frac{A_{square}}{A_{circle}}$?

Answer: $\frac{8r}{2\pi r} = \frac{4r^2}{\pi r^2}$ Yes!

Is that a coincidence?

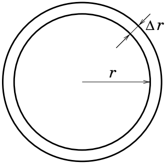
Why $\dfrac{P_{square}}{P_{circle}} = \dfrac{A_{square}}{A_{circle}}$?

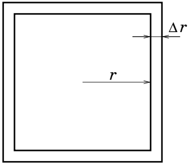
Equivalently:

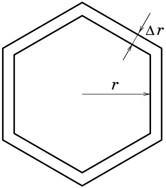
$$\frac{A_{circle}}{P_{circle}} = \frac{A_{square}}{P_{square}} = \frac{A_{hexagon}}{P_{hexagon}}$$

$$\frac{\pi r^2}{2\pi r} = \frac{4r^2}{8r} = \frac{?}{?}$$

The denominator is the derivative of the numerator!

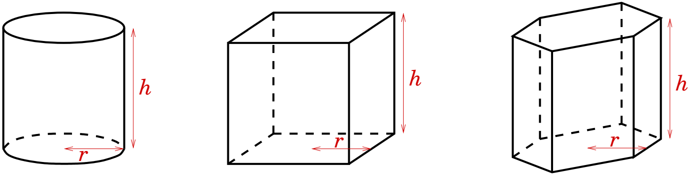






$$\frac{ar^2}{2ar} = \frac{br^2}{2br} = \frac{cr^2}{2cr}$$

Other boxes



Optimal shape: $h = 2r$

Thank you!