Math Field Day

Leap Frog Relay 9-12 sample questions

(taken from previous years)

Note: the problems given below are examples of problems given in previous years. They do not cover all the topics that can occur on the contest this year. They are only intended to give you a rough idea of the difficulty of the problems that may be given.

1. Let $A = \log_2 4 + \log_4 8 + \log_8 16 + \ldots + \log_{2^{2003}} 2^{2004}$ and $B = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2003}$. Then $A - B =$
   - (a) $\frac{1}{2003}$
   - (b) 2004
   - (c) $\frac{1}{2004}$
   - (d) 2003
   - (e) None of these
2. The average of 19 test scores is 75. Another test score comes in and it is 90. The average of the combined 20 test scores is then
   - (a) 75.8
   - (b) 75.85
   - (c) 75.9
   - (d) 75.95
   - (e) None of these
3. (Fill in the blank.) A __________ condition for a natural number N to be divisible by 6 is that N is divisible by 2 or N is divisible by 3. (The "or" here is inclusive.)
   - (a) necessary and sufficient
   - (b) necessary but not sufficient
   - (c) sufficient, but not necessary
   - (d) neither necessary nor sufficient
   - (e) None of these

4. What is the radius of the circumscribing circle to the triangle whose respective side lengths are 6, 8, and 10 inches?

   (a) $2+\sqrt{10}$
   (b) $3+\sqrt{5}$
   (c) $\sqrt{27}$
   (d) 5
   (e) None of these

5. Suppose $f(x)$ is a function such that $f(2)=2$ and $xf(x)=f(x^2-x+1)$ for every x. Then $f(3)=$

   (a) 0
   (b) 3
   (c) 6
   (d) 9
   (e) None of these

6. The number of shells in a box is a three digit number N. If the shells are divided into groups of 3, 2 shells are left over; if divided into groups of 5, 3 shells are left over; and if divided into groups of 7, 2 shells are left over. Assuming N is the smallest possible three digit number that satisfies the above conditions, then the sum of the digits of N is

   (a) 10
   (b) 11
   (c) 8
   (d) 12
   (e) None of these

7. $\sqrt{3+\sqrt{2}}-\sqrt{3-\sqrt{2}}=$

   (a) $\sqrt{\sqrt{7}-\sqrt{2}}$
   (b) $\sqrt{6-4\sqrt{2}}$
   (c) $\sqrt{2-\sqrt{2}}$
   (d) $\sqrt{6-2\sqrt{7}}$
   (e) None of these

8. Assume θ satisfies the inequalities $0^\circ \leq \theta^\circ \leq 360^\circ$ and $-1 \leq \cos\theta^\circ \leq 0 \leq \sin\theta^\circ \leq 1$. Then the maximum possible value for $\cos\frac{\theta^\circ}{2}$ is

   (a) 0
   (b) $\frac{-1}{\sqrt{2}}$
   (c) $\frac{1}{\sqrt{2}}$
   (d) $\frac{-1}{2}$
   (e) None of these

9. The area enclosed by a triangle whose side lengths are 2, 3, and 4 is

   (a) $\frac{3\sqrt{3}}{2}$
   (b) $2\sqrt{2}$
   (c) 3
   (d) $\sqrt{2}+\sqrt{3}$
   (e) None of these

10. The smallest prime number that divides $2^{2003}+3^{2003}$ is

   (a) 7
   (b) 19
   (c) $2^{2003}+1$
   (d) $3^{2003}+1$
   (e) None of these

11. The sum of the four distinct complex roots to the polynomial $x^4+2x^3+3x^2+4x+5$ is

   (a) 4
   (b) $\sqrt{5}$
   (c) i
   (d) $4i$
   (e) None of these

12. If $\log_a b = 2$ and $\log_b c = 3$, then $\log_{ab} bc =$

(a) $\dfrac{5}{2}$

(b) $\dfrac{7}{3}$

(c) 5

(d) $\dfrac{8}{3}$

(e) None of these

13. Suppose the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ can be inscribed in the diamond shape whose vertices are $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$. Then $a^2 + b^2 =$

(a) 1

(b) a^2b^2

(c) $\frac{1}{ab}$

(d) a^4b^4

(e) None of these

14. Determine the sum of the digits of the sum of the digits of the sum of the digits of 100!. $(100! = 1 \times 2 \times 3 \times 4 \times \ldots \times 100.)$

(a) 9

(b) 3

(c) 6

(d) 18

(e) None of these

15. Suppose r_1, r_2, r_3 are the three roots to the cubic equation $2002 + 2003x + 2004x^2 + 2005x^3 = 0$. Then $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} =$

(a) $-\dfrac{2004}{2003}$

(b) $-\dfrac{2003}{2002}$

(c) $-\dfrac{2004}{2005}$

(d) $-\dfrac{2005}{2004}$

(e) None of these

16. The value of the infinite continued fraction $1 + \cfrac{1}{1 + \frac{3}{1+\frac{1}{1+\frac{3}{1+\dots}}}}$ is

(a) $\dfrac{1+\sqrt{5}}{2}$

(b) $\sqrt{3}$

(c) $\dfrac{-1+\sqrt{13}}{2}$

(d) 3

(e) None of these

17. If $\sin\theta^o = \dfrac{1}{3}$ and $90^o < \theta^o < 180^o$, then $\dfrac{\cos\frac{\theta^o}{2} - \sin\frac{\theta^o}{2}}{\cos\frac{\theta^o}{2} + \sin\frac{\theta^o}{2}} =$

(a) $\dfrac{-\sqrt{2}}{2}$

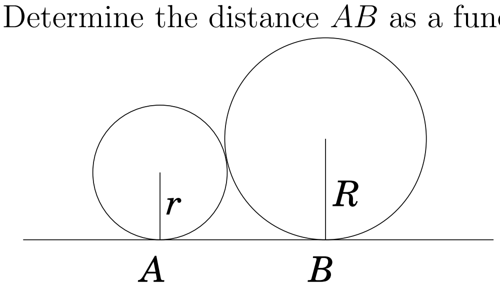
(b) $\dfrac{\sqrt{2}}{2}$

(c) $\dfrac{-\sqrt{3}}{3}$

(d) $\dfrac{\sqrt{3}}{3}$

(e) None of these

18. The two circles pictured are mutually tangent and tangent to the line $\bar{AB}$ at the respective points A and B. Determine the distance AB as a function of r and R.



(a) $\sqrt{r} + \sqrt{R}$

(b) $\sqrt{r+R}$

(c) $\dfrac{r+R}{2}$

(d) $2\sqrt{rR}$

(e) None of these

19. Among all real number pairs (x, y) that satisfy $x^2 + x + y^2 + y = 1$, find the largest possible value of $x + y$.

    (a) $\sqrt{2} - 1$
    (b) 1
    (c) $\sqrt{3} - 1$
    (d) $\sqrt{3}$
    (e) None of these

20. The value of the sum $\displaystyle\sum_{k=0}^{2004} \cos \frac{k\pi}{2004}$ lies in the interval

    (a) between -0.5 and 0.5
    (b) less than -0.5
    (c) between 0.5 and 1.5
    (d) more than 1.5
    (e) None of these

21. You select N integers, $x_1, x_2, \ldots, x_N$, at random. What is the smallest value of N that will insure that at least one difference $x_i^2 - x_j^2$, $i \neq j$, is divisible by 5?

    (a) 10
    (b) 5
    (c) 4
    (d) 6
    (e) None of these