Leap Frog Relay I Solutions–2005

No calculators allowed
Correct Answer = 4, Incorrect Answer = −1, Blank = 0

1. If 5 circles balance 6 triangles, and 1 square balances a circle and a triangle, how many squares balance 11 triangles?

   (a) 7 (b) 4

   (c) 6 (d) 5

   (e) None of these

   Solution. (d) Use C for Circle, T for Triangle and S for Square. We are given CCCCC = TTTTTT and S = CT. So

$$\begin{aligned}\text{TTTTTTTTTTT} &= \text{(TTTTT)(TTTTTT)}\\ &= \text{(TTTTT)(CCCCC)}\\ &= \text{(CT)(CT)(CT)(CT)(CT)}\\ &= \text{SSSSS.}\end{aligned}$$

   This means 5 Squares will balance 11 Triangles.

2. Suppose the line $ax + by = 1$ and the ellipse $ax^2 + by^2 = 1$ meet at exactly one point in the plane, and that a and b are both positive. Then $a + b =$ ________.

   (a) $1/\sqrt{2}$ (b) 1

   (c) $\sqrt{2}$ (d) 1/2

   (e) None of these

Solution. **(b)** Solve for y from the equation $ax + by = 1$ to get $y = (1 - ax)/b$. Substitute this into the equation $ax^2 + by^2 = 1$ and simplify to get the quadratic in x,

$$(ab + a^2)x^2 - 2ax + (1 - b) = 0.$$

This has only one solution when its discriminant is equal to zero:

$$4a^2 - 4(ab + a^2)(1 - b) = 0.$$

This simplifies to the equation

$$a + b = 1.$$

3. A sheep can clear a field of grass in one day and a cow can clear the same field in half a day. How long does it take for the the combined efforts of the sheep and cow to clear the field?

(a) 1/6 of a day

(b) 1/4 of a day

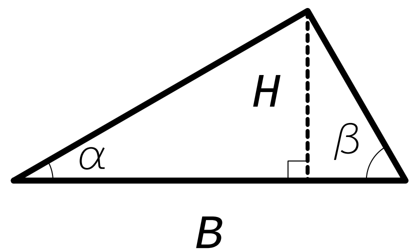
(c) 1/3 of a day

(d) 1/5 of a day

(e) None of these

Solution. **(c)** Since the cow can eat at twice the rate of the sheep, at any given point in time, the cow has eaten twice as much as the sheep. So when done, the cow has consumed 2/3 of the field and the sheep 1/3. How long does it take the sheep to eat 1/3 of the field? Since the sheep eats 1 field per day, it can eat 1/3 of the field in 1/3 of the day. So the answer is the cow and sheep, in concert, will eat the field in 1/3 of a day.

4. The height H of the triangle below, as a function of angle measures α, β, and base length B, is ________.



(a) $B(\tan\alpha + \tan\beta)$

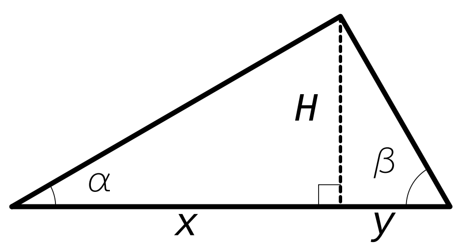
(b) $B(\cot\alpha + \cot\beta)$

(c) $\dfrac{B}{\cot\alpha + \cot\beta}$

(d) $\dfrac{B}{\tan\alpha + \tan\beta}$

(e) None of these

Solution. **(c)** The base naturally splits into two pieces, $B = x + y$, as pictured.



First note that $\tan\alpha = H/x$ and $\tan\beta = H/y$. From here, we can solve for $x = H\cot\alpha$ and $y = H\cot\beta$. Thus

$$B = x + y = H(\cot\alpha + \cot\beta).$$

Solve for H from this last equation.

$$H = \frac{B}{\cot\alpha + \cot\beta}.$$

5. $\prod_{j=2}^{2005} \log_j(j+1) =$ ________.

(a) $\log_{2005} 2006$

(b) $\log_2 2006$

(c) $\log_{2005} 2004$

(d) $\log_{2006} 2$

(e) None of these

Solution. **(b)** Use the identity

$$\log_b a = \frac{\log_c a}{\log_c b}$$

to reform the terms in the product, and cancel

$$\begin{aligned}\prod_{j=2}^{2005} \log_j(j+1) &= (\log_2 3)(\log_3 4)(\log_4 5)\cdots(\log_{2005} 2006) \\ &= (\log_2 3)\frac{\log_2 4}{\log_2 3}\frac{\log_2 5}{\log_2 4}\cdots\frac{\log_2 2006}{\log_2 2005} \\ &= \log_2 2006.\end{aligned}$$

6. Three flies land in the bottom of a square bottomed box that is partitioned into 4 equal sized little squares with vertical partitions. Assuming that the flies are independent of one another, what is the probability that at least two of the flies will land in the same little square?

(a) $\frac{11}{16}$

(b) $\frac{9}{16}$

(c) $\frac{5}{8}$

(d) $\frac{3}{8}$

(e) None of these

Solution. **(c)** The probability that at least two of the flies will land in the same little square is equal to one minus the probability that the flies land in three different squares. The number of possible outcomes is equal to 4^3. Of these, the number of outcomes where the flies land in different squares is $4 \times 3 \times 2$, since there are 4 possible squares for fly #1 to land and hence 3 for fly #2, and then 2 for fly #3. Our probability calculation is then

$$1 - \frac{4 \times 3 \times 2}{4^3} = 1 - \frac{3}{8} = \frac{5}{8}.$$

7. What is the volume of the cube that circumscribes the sphere that circumscribes the cube that circumscribes the sphere of radius 1 inch?

(a) $9\sqrt{3}$ in^3

(b) $16\sqrt{2}$ in^3

(c) $24\sqrt{3}$ in^3

(d) $54\sqrt{2}$ in^3

(e) None of these

Solution. (c) Place everything in a 3-dimensional coordinate system where the center of the sphere-cube complex is the origin $(0, 0, 0)$. It should be easy to see that the corner of the smaller cube in the first octant has coordinates $(1, 1, 1)$, and is $\sqrt{3}$ inches from the origin. So the larger sphere has radius equal to $\sqrt{3}$ inches. This then means that the corner of the larger cube in the first octant is the intersection of the three planes $x = \sqrt{3}$, $y = \sqrt{3}$, $z = \sqrt{3}$, in otherwords, the point $(\sqrt{3}, \sqrt{3}, \sqrt{3})$. From this, we can deduce that the side lengths of the larger cube are all equal to $2\sqrt{3}$. The volume of the cube is then

$$\left(2\sqrt{3}\right)^3 = 24\sqrt{3} \text{ in}^3.$$

8. Suppose you list the odd counting numbers that are *not* multiples of 3 in order $1, 5, 7, 11, \ldots$. The one-thousandth number in the list is ________.

(a) 2999

(b) 3001

(c) 3005

(d) 3007

(e) None of these

Solution. (a) Let n_i be the ith number in the list. The first 10 values of n_i are given in the table below.

i	1	2	3	4	5	6	7	8	9	10
n_i	1	5	7	11	13	17	19	23	25	29

There is a pattern for even values of i that is apparent in the table:

$$n_{2i} = 6(i-1) + 5.$$

This formula can be proved by mathematical induction. The formula is clearly true for $i = 1$. Assuming $n_{2i} = 6(i-1)+5$, then the very next odd number is $6(i-1)+5+2 = 6i+1$ (not divisible by 3), and so the next odd number after that is $6i+1+2 = 6i+3$, a number divisible by 3. This then implies $n_{2(i+1)} = 6i+5$, completing the math induction step.

Now that we have our formula, we can use it to determine the one-thousandth number in the list

$$n_{1000} = n_{2\times 500} = 6(499) + 5 = 2999.$$

9. If

$$f(x) = \sum_{k=1}^{2005} kx^{k-1},$$

then $(1-x)^2 f(x) =$ ________.

(a) $1 - 4008x^{2005} + 2005x^{2006}$

(b) $1 - 2004x^{2005} + 2005x^{2006}$

(c) $1 + 2005x^{2006}$

(d) $1 - 2006x^{2005} + 2005x^{2006}$

(e) None of these

Solution. (d) First multiply $f(x)$ by $(1-x)$.

$$\begin{aligned}
(1-x)f(x) &= (1-x)\sum_{k=1}^{2005} kx^{k-1} \\
&= (1 + 2x + 3x^2 + \cdots + 2005x^{2004}) - \\
&\quad\ (x + 2x^2 + 3x^3 + \cdots + 2005x^{2005}) \\
&= 1 + x + x^2 + \cdots + x^{2004} - 2005x^{2005}.
\end{aligned}$$

Now multiply by the second $(1-x)$.

$$\begin{aligned}(1-x)(1+x+x^2+\cdots+x^{2004}-2005x^{2005}) &= \\ (1+x+x^2+\cdots+x^{2004}-2005x^{2005})- \\ (x+x^2+x^3+\cdots+x^{2005}-2005x^{2006}) \\ &= 1-2006x^{2005}+2005x^{2006}.\end{aligned}$$

10. The *real* number $\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}$ is actually an integer. What is its value?

(a) 3

(b) 4

(c) 5

(d) 6

(e) None of these

Solution. (b) Let $a=\sqrt[3]{2+\sqrt{-121}}$ and $b=\sqrt[3]{2-\sqrt{-121}}$. The first thing to notice is

$$\begin{aligned} ab &= \sqrt[3]{(2+\sqrt{-121})(2-\sqrt{-121})} \\ &= \sqrt[3]{4-(-121)} \\ &= \sqrt[3]{125} \\ &= 5. \end{aligned}$$

Now let $x=a+b$ and compute x^3,

$$\begin{aligned} x^3 &= (a+b)^3 \\ &= a^3+3a^2b+3ab^2+b^3 \\ &= (2+\sqrt{-121})+3ab(a+b)+(2-\sqrt{-121}) \\ &= 4+3\cdot 5x \\ &= 4+15x. \end{aligned}$$

So $a+b$ is a root of the polynomial $x^3-15x-4$. It is easy to see that 4 is a root, and long division gives us

$$x^3-15x-4=(x-4)(x^2+4x+1).$$

The roots of the second factor are $-2\pm\sqrt{3}$, which are not integers. So $a+b=4$.