Leap Frog Relay II Solutions–2005

No calculators allowed
Correct Answer = 4, Incorrect Answer = −1, Blank = 0

1. Go forward 100 miles, backwards 99 miles, forward 98 miles, backwards 97 miles, etc. ending with your final backwards 1 mile, and stop! How far are you from your original starting point?

   (a) 50 miles (b) 51 miles

   (c) 49 miles (d) 48 miles

   (e) None of these

   Solution. (a) Group the instructions in pairs (100, 99), (98, 97), (96, 95), ... , (2, 1). The end result of each pair is a forward motion of 1 mile. There are 50 pairs, so you have moved 50 miles from your original position.

2. The currency in Freedonia has three denominations, the blip, the blorp, and the blurp. This unusual currency follows the "8 to 5 rule". This means, 8 blips are equal in value to 5 blorps and 8 blorps are equal to 5 blurps. Suppose Harpo buys a spaghetti burrito for 14 blorps. He gives the cashier a 20-blurp note, and the cashier gives Harpo his change in blips and blurps. What did Harpo get back?

   (a) 5 blurps and 15 blips (b) 5 blurps and 16 blips

   (c) 5 blurps and 17 blips (d) 5 blurps and 18 blips

   (e) None of these

Solution. (b) He gets 5 blurps and 16 blips in change.

$$\begin{aligned} 20 \text{ blurps} - 14 \text{ blorps} &= 5 \text{ blurps} + 15 \text{ blurps} - 14 \text{ blorps} \\ &= 5 \text{ blurps} + 24 \text{ blorps} - 14 \text{ blorps} \\ &= 5 \text{ blurps} + 10 \text{ blorps} \\ &= 5 \text{ blurps} + 16 \text{ blips} \end{aligned}$$

3. At noon, the hour and minute hand of a clock are pointing in the same direction. At what exact time will this happen next?

(a) $1:05:27\frac{1}{11}$

(b) $1:05:27\frac{2}{11}$

(c) $1:05:27\frac{3}{11}$

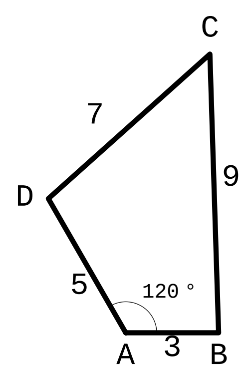
(d) $1:05:27\frac{4}{11}$

(e) None of these

Solution. (c) After 12 o'clock, the hour hand meets the minute hand exactly 11 times during the 12 hours before the next 12 o'clock. What's more, the exact same time elapses between each meeting (look at it from the point of view of the hour hand). So, the next time after 12 when this occurs is $12/11 = 1\frac{1}{11}$ hours later. The fraction $\frac{1}{11}$ of an hour is $5\frac{5}{11}$ minutes. Since $\frac{1}{11}$ of a minute is $5\frac{5}{11}$ seconds, then $\frac{5}{11}$ of a minute is $(5\frac{5}{11}) \times 5 = 27\frac{3}{11}$ seconds. Putting this all together, the exact time is

$$1:05:27:\frac{3}{11}.$$

4. Determine the area enclosed by the quadrilateral pictured below.



(a) $\frac{2}{3}(5\sqrt{3}+3\sqrt{115})$

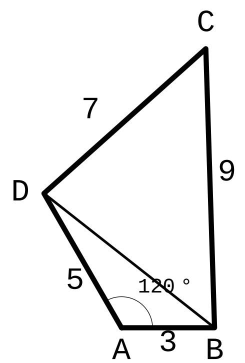
(b) $\frac{5}{7}(5\sqrt{3}+3\sqrt{115})$

(c) $\frac{3}{5}(5\sqrt{3}+3\sqrt{115})$

(d) $\frac{3}{4}(5\sqrt{3}+3\sqrt{115})$

(e) None of these

Solution. **(d)** Draw the segment BD.



By the Law of Cosines,

$$\begin{aligned} |BD|^2 &= 3^2+5^2-2\times 3\times 5\times\cos(120°) \\ &= 34-30\cos(120°) \\ &= 34-30(-1/2) \\ &= 49. \end{aligned}$$

So this tells us $BD = 7$. From here you can use Heron's formula to compute the area enclosed by a triangle given that the three side lengths are a, b, c,

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = (a+b+c)/2$, the *semi-perimiter*. For $\triangle ABD$, $s = (3+5+7)/2 = 15/2$,

$$\begin{aligned} \triangle ABD \text{ Area} &= \sqrt{\frac{15}{2}\left(\frac{15}{2}-3\right)\left(\frac{15}{2}-5\right)\left(\frac{15}{2}-7\right)} \\ &= \frac{15\sqrt{3}}{4}. \end{aligned}$$

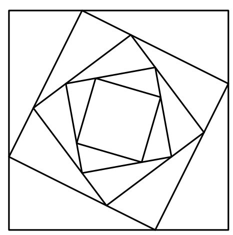
As for $\triangle BDC$, $s = (7+7+9)/2 = 23/2$, and

$$\begin{aligned}\triangle BDC \text{ Area} &= \sqrt{\frac{23}{2}\left(\frac{23}{2}-7\right)\left(\frac{23}{2}-7\right)\left(\frac{23}{2}-9\right)} \\ &= \frac{9\sqrt{115}}{4}.\end{aligned}$$

So the combined area is the sum,

$$\frac{15\sqrt{3}}{4} + \frac{9\sqrt{115}}{4} = \frac{3}{4}(5\sqrt{3} + 3\sqrt{115}).$$

5. In the figure below, each square touches its inscribed square so that the circumscribing square's side is cut in a ratio of 2:1. The ratio of the smallest square's area to the largest square's area is ______.



(a) $\dfrac{625}{6561}$

(b) $\dfrac{16}{81}$

(c) $\dfrac{1}{16}$

(d) $\dfrac{1}{9}$

(e) None of these

Solution. **(a)** Consider one pair of mutually circumscribing/inscribing squares. Let a be the length of the smaller square side and b the length of the larger square side. By looking at the right triangle formed whose hypotenuse is the smaller square side and whose two legs are formed from a pair of adjacent sides of the larger square, we can use the Pythagorean Theorem to get the equation

$$\left(\frac{b}{3}\right)^2 + \left(\frac{2b}{3}\right)^2 = a^2.$$

Solve for a^2,

$$a^2 = \frac{5}{9}b^2.$$

This tells us that the ratio of the smaller square area to the larger square area is $\frac{5}{9}$. Thus, the ratio of the smallest square area to the largest square area is

$$\left(\frac{5}{9}\right)^4 = \frac{625}{6561}.$$

6. You have 30 students in a room. Each student shakes the hand of each of the other students. (No student shakes another student's hand more than once, and no student shakes his own hand.) How many handshakes were there?

(a) 406

(b) 60

(c) 2^{29}

(d) $2^{30} - 1$

(e) None of these

Solution. (e) Label the students $S_1, S_2, \ldots, S_{30}$. The student labeled S_1 shakes 29 hands. Student S_2 would then have shaken 28 hands not including his handshake with student S_1 which has already been counted. We now work our way down, getting the total number of hand shakes,

$$\text{Hand Shakes} = 29 + 28 + 27 + \cdots + 2 + 1.$$

This is an arithmetic sum, which can be computed as

$$\frac{30 \times 29}{2} = 435.$$

7. A bookworm starts eating the front cover of volume 1 of the 10 volume Encyclopedia of Composting, and ends when he reaches the last page of volume 10. Assuming the books are on the bookshelf in standard order (increasing volumes left to right), the worm eats through the volumes in standard order, each volume is 2 inches thick, including covers, and each cover is one-tenth of an inch thick, how far, in inches, did the bookworm eat his way through?

(a) 17 inches

(b) 16 inches

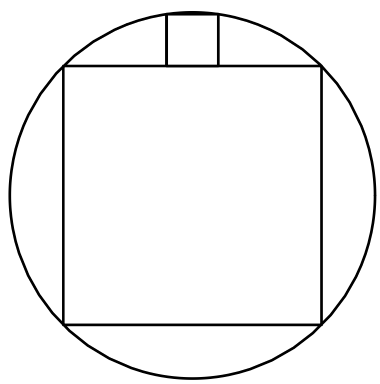
(c) 16.5 inches

(d) 16.2 inches

(e) None of these

Solution. **(d)** The worm only eats his way through one cover (the front) of volume 1 to get to volume 2, covering 1/10 inch. Volume 2 through 9 are $2 \times 8 = 16$ inches. Finally, he chews through one cover (the back) of volume 10 to get to the last page, for an additional 1/10 inch. Adding these together, $1/10 + 16 + 1/10 = 16.2$ inches.

8. In the figure below, the larger square is inscribed in the circle and the smaller square is inscribed between the larger square and the circle. Assuming the larger square's side length is 2 inches, what is the side length of the smaller square?



(a) 0.5 inches

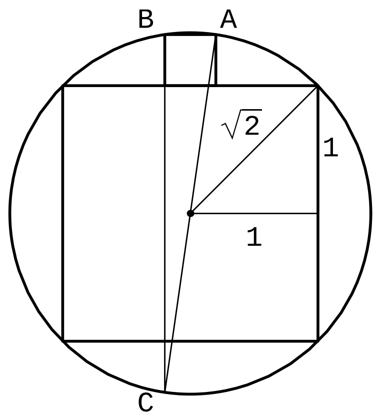
(b) $\frac{1}{2\sqrt{2}}$ inches

(c) $\frac{1}{\sqrt{5}}$ inches

(d) 0.45 inches

(e) None of these

Solution. **(e)** It is easy to see, from the Pythagorean Theorem, that the radius of the circle is $\sqrt{2}$ inches (see figure below).



Now, extend lines as in the above figure. Since $\triangle ABC$ is a right triangle, the segment $\overline{AC}$ goes through the center of the circle, and hence $AC = 2\sqrt{2}$ inches. Let the side length of the smaller square be s inches. Then, $AB = s$ and $BC = 2s + 2$. Now apply the Pythagorean Theorem on $\triangle ABC$ to get

$$(2s + 2)^2 + s^2 = (2\sqrt{2})^2.$$

Expand and regroup,

$$5s^2 + 8s - 4 = 0.$$

Now use the quadratic equation to solve for s,

$$\begin{aligned} s &= \frac{-8 \pm \sqrt{8^2 - 4 \cdot 5(-4)}}{2 \cdot 5} \\ &= \frac{-8 \pm 12}{10}. \end{aligned}$$

Since s must be a positive number, we get

$$s = \frac{4}{10} = 0.4 \text{ inches.}$$

9. When converted to base 10, the infinite repeating base 3 number $0.\overline{12}_3$ is equal to ________.

(a) $\frac{1}{2}$

(b) $\frac{4}{9}$

(c) $\frac{5}{8}$

(d) $\frac{5}{9}$

(e) None of these

Solution. (c) Let N denote the base 10 representation of the number. Then

$$\begin{aligned} N &= 1\times 3^{-1} + 2\times 3^{-2} + 1\times 3^{-3} + 2\times 3^{-4} + \cdots \\ &= 3^{-1}\times(1+3^{-2}+3^{-4}+\cdots) + (2\times 3^{-2})\times(1+3^{-2}+3^{-4}+\cdots) \\ &= \left(\frac{1}{3}+\frac{2}{9}\right)\left(1+\left(\frac{1}{9}\right)+\left(\frac{1}{9}\right)^2+\cdots\right) \\ &= \left(\frac{5}{9}\right)\left(1+\left(\frac{1}{9}\right)+\left(\frac{1}{9}\right)^2+\cdots\right). \end{aligned}$$

The infinite sum is a geometric series, so

$$\begin{aligned} 1+\left(\frac{1}{9}\right)+\left(\frac{1}{9}\right)^2+\cdots &= \frac{1}{1-\frac{1}{9}} \\ &= \frac{9}{8}. \end{aligned}$$

Thus

$$\begin{aligned} N &= \frac{5}{9}\times\frac{9}{8} \\ &= \frac{5}{8}. \end{aligned}$$

10. The smallest counting number with exactly 16 divisors has digit sum equal to ________.

(a) 3 (b) 5

(c) 7 (d) 9

(e) None of these

Solution. (a) If we factor a number N as a product of distinct prime powers $N = p_1^{a_1}\times\cdots\times p_k^{a_k}$, the number of factors of N is equal to the product $(a_1+1)\times\cdots\times(a_k+1)$, which for us must be equal to 16. So we must go through all possible factorizations of 16. Also, since we want N to be as small as possible, we will always use the smallest primes available, 2, 3, 5, 7, etc., in constructing our candidates for N. There are now several cases to consider.

- $16 = (15+1)$ suggests the candidate $N = 2^{15} = 32{,}768$.
- $16 = 8 \times 2 = (7+1) \times (1+1)$ suggests the candidate $N = 2^7 \times 3 = 384$.
- $16 = 4 \times 4 = (3+1) \times (3+1)$ suggests the candidate $N = 2^3 \times 3^3 = 216$.
- $16 = 4 \times 2 \times 2 = (3+1) \times (1+1) \times (1+1)$ suggests the candidate $N = 2^3 \times 3 \times 5 = 120$.
- $16 = 2 \times 2 \times 2 \times 2 = (1+1) \times (1+1) \times (1+1) \times (1+1)$ suggests the candidate $N = 2 \times 3 \times 5 \times 7 = 210$.

The smallest among our candidates is $N = 120$, whose digit sum is equal to 3.