Math 141 **Summer 2000**

Practice Final

1. Find $f'(x)$ and state its domain.

   (a)
   $$f(x) = \frac{x}{(2+3x)^{\frac{1}{3}}}$$

   (b)
   $$f(x) = \sin(\cos(\ln x))$$

2. Find an equation of the tangent line to

   (a) $y = x + \sqrt{x}$ at $(1, 2)$

   (b) $y = \tan x$ at $(0, 0)$

   (c) $y = 2^{x-1}$ at $(2, 2)$

3. The position function of a particle is given by $s(t) = t^3 - 3t^2 - 1, \;\; t \geq 0$, where t is measured in seconds and s in meters.

   (a) Find the velocity at time t.

   (b) What is the velocity after 3 seconds?

   (c) When is the particle at rest?

   (d) When is the particle moving in the positive direction?

   (e) Find the total distance travelled during the first 3 seconds.

   (f) Find the average velocity over the time interval $[0, 3]$.

   (g) Find the average speed over the time interval $[0, 3]$.

4. Show that the equation $3x^5 - 10x^3 + 15x - 8 = 0$ has only one real root.

5. Find y' by implicit differentiation.

   (a) $x = \cos y$

   (b) $\sqrt{x+y} + \sqrt{xy} = 6$

6. Let $P(x) = ax^2 + bx + c$.

   (a) Find $P'(x)$, $P''(x)$, and $P'''(x)$.

   (b) Notice that $P''(x)$ is a constant. Find $P(x)$ such that $P''(x) = 1$ for all x, and $P'(0) = 2$ and $P(0) = 3$.

7. Let $y = \sin(2t)$.

   (a) Find Dy, D^2y, D^3y, D^4y, D^5y.

   (b) What is $D^{100}y$? $D^{101}y$?

8. Use logarithmic differentiation to find the derivative of

   (a)
   $$f(x) = \sqrt{x}e^{x^2}(x^2+1)^{10}$$

   (b)
   $$g(x) = x^{\sin x}$$

9. (a) Find the diffferential of $y = \frac{x+1}{x-1}$

   (b) Evaluate dy for $x = 2$ and $dx = .001$.

   (c) Use (a) and (b) to estimate $\frac{3.001}{1.001}$

10. Let $f(x) = x^3 - 12x + 1$.

    (a) Find the intervals on which $f(x)$ is increasing or decreasing.

    (b) Find the local and absolute maximum and minimum values of $f(x)$.

    (c) Find the intervals of concavity and the inflection points.

    (d) Use (a)-(c) to sketch the graph of $f(x)$.

11. Do Problem 10 for $f(x) = xe^x$.

12. Use l'Hospital's Rule to find the limits.

$$\lim_{x\to\infty} \frac{e^x}{x^2}, \quad \lim_{x\to 0} \frac{\sin x - x}{x^3}, \quad \lim_{x\to\infty} xe^{1/x} - x, \quad \lim_{x\to\infty} x^{1/x}$$

13. State the Principle of Mathematical Induction. Give an example of a statement that can be proved using this principle.

Supplementary problems

Unless you have lots of free time, you probably won't have a chance to attempt all the problems in this list. Make sure you try some from each section which seems difficult for you. Only odd-numbered problems appear in the list below, since the answers are in the back of the book. However, if you want even more practice, you can try similar even-numbered problems, too.

- **Chapter 1 Review Exercises** 9, 13, 23
- **Chapter 2 Review Exercises** 17, 21, 35, 37, 51
- **3.1** 9, 27, 53, 57
- **3.2** 11, 25, 31
- **3.3** 9, 13, 15
- **3.4** 11, 19, 23, 41
- **3.5** 9, 41, 45, 57
- **3.6** 1, 7, 27, 41
- **3.7** 3, 11, 29, 57
- **3.8** 9, 27, 35, 47
- **3.10** 3, 7, 11
- **3.11** 7, 17, 23, 31
- **Chapter 3 Review Exercises** 17, 21, 37, 51, 53, 81, 89, 99
- **4.1** 5, 7, 49, 57
- **4.2** 1, 17, 27, 31
- **4.3** 7, 15, 27, 41
- **4.4** 13, 55, 63, 71
- **Chapter 4 Review Exercises** 3, 11, 45

As always, please feel free to ask for help if you need it.

Answers

1. (a)

$$f'(x) = \frac{2 + \frac{8}{3}x}{(2+3x)^{4/3}}$$

Domain $= \mathbb{R} - \{-\frac{2}{3}\} = (-\infty, -\frac{2}{3})\mathrm{U}(-\frac{2}{3}, +\infty)$

(b)

$$f'(x) = -\frac{\cos(\cos(\ln x))\sin(\ln x)}{x}$$

Domain $= (0, +\infty)$

2. (a) $y = \frac{3}{2}x + \frac{1}{2}$

(b) $y = x$

(c) $y = (2\ln 2)x + (2 - 4\ln 2)$

3. (a) $v(t) = 3t^2 - 6t$

(b) $v(3) = 9$

(c) $t = 0$ and $t = 2$ (when $v(t) = 0$)

(d) $t < 0$ or $t > 2$ (when $v(t) > 0$)

(e) 8

(f) 0

(g) 8/3

4. Hint: first, show that the equation has at least one root. Second, show that the derivative of $f(x) = 3x^5 - 10x^3 + 15x - 8$ is nonnegative for all x. Deduce that the given equation can not have more than one real root.

5. (a)

$$y' = -\frac{1}{\sin y}$$

(b)

$$y' = -\frac{\sqrt{xy} + y\sqrt{x+y}}{\sqrt{xy} + x\sqrt{x+y}}$$

6. (a) $P'(x) = 2ax + b, \quad P''(x) = 2a, \quad P'''(x) = 0.$

   (b) $\frac{1}{2}x^2 + 2x + 3$

7. (a) $Dy = 2\cos(2x), \quad D^2y = -4\sin(2x), \quad D^3y = -8\cos(2x),$
   $D^4y = 16\sin(2x), \quad D^5y = 32\cos(2x)$

   (b) $D^{100}y = 2^{100}\sin(2x), \ D^{101}y = 2^{101}\cos(2x)$

8. (a)

$$f'(x) = \sqrt{x}e^{x^2}(x^2+1)^{10}(\frac{1}{2x} + 2x + \frac{20x}{x^2+1})$$

   (b)

$$g'(x) = x^{\sin x}(\cos x \ln x + \frac{\sin x}{x})$$

9. (a)

$$dy = -\frac{2}{(x-1)^2}dx$$

   (b) $dy = -.002$

   (c) $\frac{3.001}{1.001} = y(2.001) \qquad y(2) + dy = 2.998$

10. (a) $f(x)$ is increasing on $(-\infty, -2)$ and $(2, +\infty)$ (when $f'(x) > 0$) and decreasing on $(-2, 2)$ (when $f'(x) < 0$).

   (b) $x = -2$ is a local maximum, $f(-2) = 17$.
   $x = 2$ is a local minimum, $f(2) = -15$.
   No absolute maximim or minimum.

   (c) CU on $(0, +\infty)$, CD on $(-\infty, 0)$

   (d)

11. (a) increasing on $(-1, +\infty)$ and decreasing on $(-\infty, -1)$.
(b) $x = -1$ is a local and absolute minimum, $f(-1) = -\frac{1}{e}$.
No absolute or local maximum.
(c) CU on $(-2, +\infty)$, CD on $(-\infty, -2)$
(d)

12. $+\infty$, $-\frac{1}{6}$, 1, 1

13. Principle of Mathematical Induction:
Let S_n be a statement about the positive integer n. Suppose that:
1.S_1 is true.
2.S_{k+1} is true whenever S_k is true.
Then S_n is true for all natural numbers n.

Example: Let S_n be the fomula

$$1 + 2 + ... + n = \frac{n(n+1)}{2}$$

This formula can be proved by Mathematical Induction as follows.
1. S_1 says that $1 = \frac{1(1+1)}{2}$. This is true.
2. Suppose that S_k is true, i.e. $1 + 2 + ... + k = \frac{k(k+1)}{2}$.
Then

$$1 + 2 + ... + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

That is,

$$1 + 2 + ... + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$$

Thus we proved the formula S_{k+1}.