Hoover High School Math League

Bases other than 10 — Theory

Integers

Any nonnegative integer N (in base 10) with digits a_n, a_{n-1}, $\ldots$, a_2, a_1, a_0, can be written as a sum of multiples of powers of 10:

$$\begin{aligned} N &= a_n a_{n-1} \ldots a_1 a_0 = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \ldots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0 \\ &= a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \ldots + a_2 \cdot 10^2 + a_1 \cdot 10^1 + a_0 \cdot 10^0. \end{aligned}$$

Similarly, for any natural number b, we can write the number N as a sum of multiples of powers of b (with coefficients less than b):

$$N = c_k \cdot b^k + c_{k-1} \cdot b^{k-1} + \ldots + c_2 \cdot b^2 + c_1 \cdot b^1 + c_0 \cdot b^0.$$

Then we say that $(c_k c_{k-1} \ldots c_2 c_1 c_0)_b$ is the base b representation of the number N.

Example 1. Let $N = 100$.

For $b = 2$, the coefficients must be less than 2, so they can only be either 0 or 1. Since

$$100 = 1 \cdot 2^6 + 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0,$$

$$100 = 1100100_2.$$

The base 2 representation is called the *binary* representation.

For $b = 3$, the coefficients must be less than 3. Since

$$100 = 1 \cdot 3^4 + 0 \cdot 3^3 + 2 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0,$$

the base 3 representation of 100 is

$$100 = 10201_3.$$

Example 2. To find what number (in base 10) is represented by 1234_5 (base 5), we compute the corresponding sum:

$$1234_5 = 1 \cdot 5^3 + 2 \cdot 5^2 + 3 \cdot 5^1 + 4 \cdot 5^0 = 1 \cdot 125 + 2 \cdot 25 + 3 \cdot 5 + 4 \cdot 1 = 194.$$

Remark. If base $b > 10$, letters A, B, C, etc. are used for ”digits” 10, 11, 12, etc. respectively.

Example 3. In base 15, the number $A3D_{15}$ represents $10 \cdot 15^2 + 3 \cdot 15 + 13 = 2308$.

Decimals

The idea is similar to that for integers. In base 10, a decimal can be written as

$$0.a_1a_2a_3\ldots = a_1 \cdot \frac{1}{10} + a_1 \cdot \frac{1}{10^2} + a_1 \cdot \frac{1}{10^3} + \ldots.$$

For a decimal represented in base b, we have

$$(0.c_1c_2c_3\ldots)_b = c_1 \cdot \frac{1}{b} + c_1 \cdot \frac{1}{b^2} + c_1 \cdot \frac{1}{b^3} + \ldots.$$

Example. The number 0.1023_5 represents

$$0.1023_5 = 1 \cdot \frac{1}{5} + 0 \cdot \frac{1}{5^2} + 2 \cdot \frac{1}{5^3} + 3 \cdot \frac{1}{5^4} = \frac{1}{5} + \frac{2}{5^3} + \frac{3}{5^4} = \frac{5^3 + 2 \cdot 5 + 3}{5^4} = \frac{125 + 10 + 3}{625} = \frac{138}{625} = 0.2208$$