MATH 105

The final exam is on Friday, December 15, 10:30 AM - 12:30 PM, in BT 1688.

Sample Final Exam - Solutions

1. Evaluate: $\dfrac{6! \cdot 6^{1.5}}{2!\sqrt{24}} = \dfrac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 6^{1.5}}{2\sqrt{4}\sqrt{6}} = \dfrac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 6^{1.5}}{2 \cdot 26^{0.5}} = 2 \cdot 3 \cdot 5 \cdot 6 \cdot 6 = 1080$

2. Solve the inequality:

   (a) $3x + 6 < 5 - x$
   $4x < -1$
   $x < -\dfrac{1}{4}$
   Ans: $\left(-\infty, -\dfrac{1}{4}\right)$

   (b) $6x - 8 > x^2$
   $x^2 - 6x + 8 < 0$
   $(x-2)(x-4) < 0$
   $2 < x < 4$
   Ans: $(2, 4)$

3. Find an equation of the line through $P(2, -4)$ and $Q(-1, 5)$.
   slope $= \dfrac{5+4}{-1-2} = -3$
   Equation: $y + 4 = -3(x - 2)$
   $y = -3x + 6 - 4$
   $y = -3x + 2$

4. Let $f(x) = 8x - 1$, $g(x) = \sqrt{x-2}$.

   (a) Find $f \circ g(x)$ and its domain.
   $f \circ g(x) = 8\sqrt{x-2} - 1$
   $f \circ g(x)$ is defined when $x - 2 \geq 0$, i.e. $x \geq 2$.
   Domain: $[2, +\infty)$

(b) Find $g \circ f(x)$ and its domain.

$g \circ f(x) = \sqrt{8x - 1 - 2} = \sqrt{8x - 3}$

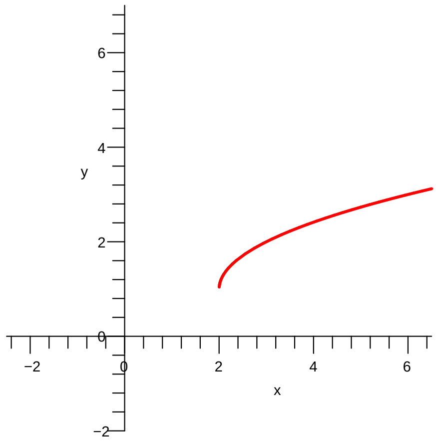
$g \circ f(x)$ is defined when $8x - 3 \geq 0$, i.e. $x \geq \frac{3}{8}$.

Domain: $\left[\frac{3}{8}, +\infty\right)$

5. Sketch the graph of the function:

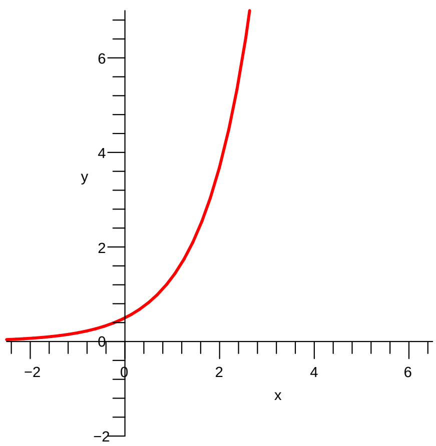
(a) $f(x) = \sqrt{x - 2} + 1$

Shift the graph of $y = \sqrt{x}$ two units to the right and one unit upward:

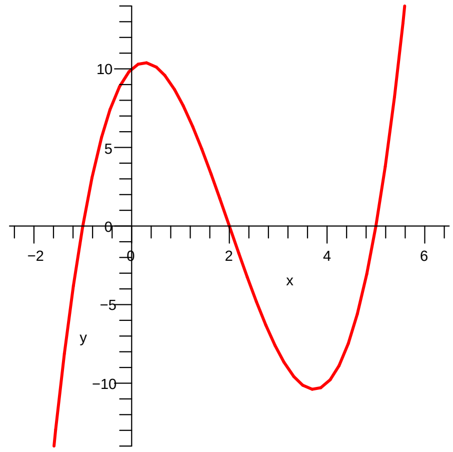


(b) $g(x) = \frac{e^x}{2}$

Compress the graph of $y = e^x$ by a factor of 2 vertically:



(c) $h(x) = (x+1)(x-2)(x-5)$
This is a cubic polynomial with x-intercepts -1, 2, and 5:



6. Simplify:

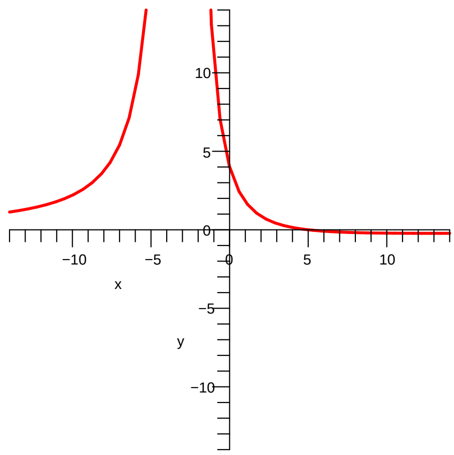
(a) $\log_5 \sqrt[3]{5} = \log_5 5^{\frac{1}{3}} = \dfrac{1}{3}\log_5 5 = \dfrac{1}{3}$

(b) $\sin(\pi) - 3\cos\left(\dfrac{\pi}{6}\right) = 0 - 3 \cdot \dfrac{\sqrt{3}}{2} = -\dfrac{3\sqrt{3}}{2}$

7. Sketch the graph and find an equation of a rational function f that satisfies the folloing four conditions:

- f has a vertical asymptote $x = -3$
- f has a horizontal asymptote $y = 0$
- 5 is an x-intercept of f
- 4 is a y-intercept of f

Exapmle: $f(x) = \dfrac{-36(x-5)}{5(x+3)^2}$



(Note: there are many such functions.)

8. Solve the equation: $\ln 3^{(x^2)} = 5$

$x^2 \ln 3 = 5$

$x^2 = \dfrac{5}{\ln 3}$

$x = \pm\sqrt{\dfrac{5}{\ln 3}}$

9. A conical paper cup is constructed by removing a sector from a circle of radius 5 inches and attaching edge OA to OB (see the figure). Find angle AOB so that the cup has a depth of 4 inches.

By the Pythagorean theorem, the radius of the top of the cup is $\sqrt{5^2 - 4^2} = 3$ (inches). Therefore the circumference of the top is 6π. Then $6\pi = 5\angle AOB$, so $\angle AOB = \dfrac{6\pi}{5}$ (radians).

10. Find all real solutions of the equation: $\tan(2x)\cos(2x) = 1$.

$\dfrac{\sin(2x)}{\cos(2x)} \cdot \cos(2x) = 1$

$\sin(2x) = 1$

$2x = \dfrac{\pi}{2} + 2\pi k$ where k is an integer

$x = \dfrac{\pi}{4} + \pi k$ where k is an integer

11. Solve the system: $\begin{cases} x - 3y = 4 \\ -2x + 6y = 2 \end{cases}$

Dividing the second equation by 2 gives $x - 3y = -2$ which contradicts the first equation. Therefore there no solutions.

12. Evaluate: $\sum_{k=1}^{4}(k-1)(k+1) = (1-1)(1+1)+(2-1)(2+1)+(3-1)(3+1)+(4-1)(4+1) = 0 + 3 + 8 + 15 = 26$

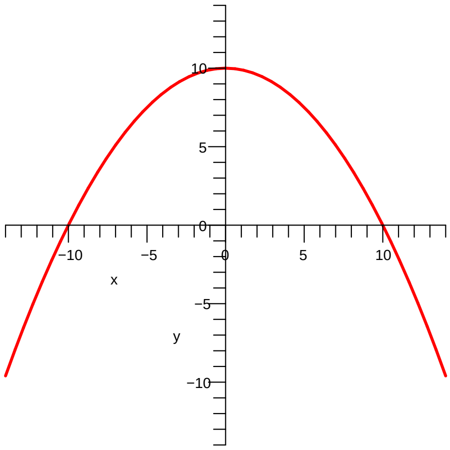
13. Express the sum in terms of summation notation: $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \ldots + \dfrac{1}{99\cdot 100} = \sum_{k=1}^{99} \dfrac{1}{k\cdot(k+1)}$

14. Sketch the graph of the equation:

(a) $10y = 100 - x^2$

$y = 10 - \dfrac{x^2}{10}$

Reflect the graph of $y = x^2$ about the x-axis, compress vertically by a factor of 10, and shift 10 units upward:



(b) $4x^2 + y^2 - 24x + 4y + 36 = 0$

$(4x^2 - 24x) + (y^2 + 4y) + 36 = 0$

$4(x^2 - 6x) + (y^2 + 4y) + 36 = 0$

$4(x^2 - 6x + 9) + (y^2 + 4y + 4) + 36 = 0 + 36 + 4$

$4(x - 3)^2 + (y + 2)^2 = 4$

$(x - 3)^2 + \dfrac{(y + 2)^2}{4} = 1$

$\dfrac{(x - 3)^2}{1^2} + \dfrac{(y + 2)^2}{2^2} = 1$

This is an ellipse with center at $(3, -2)$, $a = 1$, and $b = 2$: