MATH 110 **Lecture notes – 3**

Expressing some operations in terms of others revisited.

Recall the following from a previous lecture.

From the six operations $\neg$, $\wedge$, $\vee$, $\oplus$, $\rightarrow$, $\leftrightarrow$, some operations can be expressed in terms of others. For example,

$P \rightarrow Q \equiv \neg P \vee Q$.

Also, it can be checked using the truth tables that

$P \wedge Q \equiv \neg(\neg P \vee \neg Q)$,

$P \vee Q \equiv \neg(\neg P \wedge \neg Q)$,

$P \oplus Q \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$,

$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$.

Observations made earlier:

1. Any operation can be defined in terms of $\wedge$, $\vee$, and $\neg$.
2. Since $\wedge$ can be defined in terms of $\vee$ and $\neg$, any operation can be defined in terms of these two.
3. Since $\vee$ can be defined in terms of $\wedge$ and $\neg$, any operation can be defined in terms of these two as well.

Old questions and new answers:

1. Can $\neg$ be defined in terms of $\wedge$ and $\vee$?

   Answer: no. If this were possible, we would have an expression that contains only variables, $\wedge$, and $\vee$, and is logically equvalent to $\neg P$. However, when constructing a truth table for such an expression, we would only have the value T in the first line, where each variable has the value T. So, it is not possible to get an F in that line, therefore the expression cannot be logically equivalent to $\neg P$.
2. Can $\wedge$ and $\vee$ be defined in terms of $\rightarrow$ and $\neg$? If so, how? If not, explain why not.

   Answer: yes. Since $P \rightarrow Q \equiv \neg P \vee Q$, replacing P with $\neg P$ and eliminating the double negation, we have:

$$P \vee Q \equiv \neg P \rightarrow Q.$$

Applying negation to both sides of this gives

$$\neg(P \vee Q) \equiv \neg(\neg P \rightarrow Q).$$

Using DeMorgan's law,

$$\neg P \wedge \neg Q \equiv \neg(\neg P \rightarrow Q).$$

Finally, replace P with $\neg P$ and Q with $\neg Q$, and eliminate the double negation to obtain:

$$P \wedge Q \equiv \neg(P \rightarrow \neg Q).$$

3. Can all of these six operations be expressed in terms of just one of them? If so, which one? If not, explain why not.

   Answer: no.

   - $\neg$ is insufficient because it cannot connect two variables.
   - $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$ always will give the truth value T when each variable has the value T, therefore cannot express negation.
   - $\oplus$ will always give the value F when each variable has the value F, therefore cannot express $\leftrightarrow$.

4. Does there exist any other operation (an operation can be defined by a truth table) that could be used to define all six of the above (classical) operations?

   Answer: yes. There are two such operations, namely,

$$X \star Y = \neg(X \wedge Y)$$

   and

$$X * Y = \neg(X \vee Y).$$

   First let's show that these operations $\star$ and $*$ are the only binary operations that could possibly be capable of expressing all other operations.

   - To express negation, the value of the operation for P =T and Q =T must be F.

P	Q	P operation Q
T	T	F
T	F	
F	T	
F	F	

- To express biconditional, the value of the operation for $P =$F and $Q =$F must be T.

P	Q	P operation Q
T	T	F
T	F	
F	T	
F	F	T

- If the values of the operation at $P =$T, $Q =$F and at $P =$F, $Q =$T are T and F respectively, then the operation is equivalent to $\neg Q$, while if the values of the operation at $P =$T, $Q =$F and at $P =$F, $Q =$T are F and T respectively, then the operation is equivalent to $\neg P$. We already know that $\neg$ cannot express other operations.
- Thus these two values should be either both T or both F. In the first case we get $P \star Q$, and in the second we get $P * Q$:

P	Q	$P \star Q$
T	T	F
T	F	T
F	T	T
F	F	T

P	Q	$P * Q$
T	T	F
T	F	F
F	T	F
F	F	T

Next we will show that all other operations can be expressed in terms of $\star$.
Observe that $X \star X \equiv \neg(X \wedge X) \equiv \neg X$, so

$$\neg X \equiv X \star X.$$

Then,

$$\begin{aligned} X \wedge Y &\equiv \neg(X \star Y) \\ &\equiv (X \star Y) \star (X \star Y), \end{aligned}$$

$$
\begin{aligned}
X \vee Y &\equiv \neg((\neg X) \wedge (\neg Y)) \\
&\equiv \neg((X \star X) \wedge (Y \star Y)) \\
&\equiv \neg\big(((X \star X) \star (Y \star Y)) \star ((X \star X) \star (Y \star Y))\big) \\
&\equiv \big(((X \star X) \star (Y \star Y)) \star ((X \star X) \star (Y \star Y))\big) \star \\
&\quad \big(((X \star X) \star (Y \star Y)) \star ((X \star X) \star (Y \star Y))\big).
\end{aligned}
$$

Notice that

$$(A \star A) \star (A \star A) \equiv \neg\neg A \equiv A,$$

so the above can be simplified:

$$X \vee Y \equiv (X \star X) \star (Y \star Y).$$

Equivalently, using $X \wedge Y \equiv \neg(X \star Y)$, we could do the following:

$$
\begin{aligned}
X \vee Y &\equiv \neg((\neg X) \wedge (\neg Y)) \\
&\equiv \neg(\neg((\neg X) \star (\neg Y))) \\
&\equiv (\neg X) \star (\neg Y) \\
&\equiv (X \star X) \star (Y \star Y).
\end{aligned}
$$

Also,

$$
\begin{aligned}
X \to Y &\equiv \neg X \vee Y \\
&\equiv \neg(X \wedge \neg Y) \\
&\equiv \neg(X \wedge (Y \star Y)) \\
&\equiv \neg((X \star (Y \star Y)) \star (X \star (Y \star Y))) \\
&\equiv ((X \star (Y \star Y)) \star (X \star (Y \star Y))) \star ((X \star (Y \star Y)) \star (X \star (Y \star Y))) \\
&\equiv X \star (Y \star Y).
\end{aligned}
$$

Exercise: express $\neg$, $\wedge$, and $\vee$ in terms of $*$.