Axioms

An axiom system consists of a set of formulas (called axioms) and some rules (called rules of inference). We say that an axiom system is **sound** if every formula that is derivable from this axiom system is valid (i.e. is a tautology). An axiom system is **complete** if every valid formula (i.e. every tautology) can be derived from this axiom system.

The following is one of the sound and complete axiom systems for the classical propositional logic.

Axioms:

1. $(X \wedge Y) \to X$
2. $(X \wedge Y) \to Y$
3. $X \to (X \vee Y)$
4. $Y \to (X \vee Y)$
5. $(\neg\neg X) \to X$
6. $X \to (Y \to X)$
7. $X \to (Y \to (X \wedge Y))$
8. $((X \to Y) \wedge (X \to \neg Y)) \to \neg X$
9. $((X \to Z) \wedge (Y \to Z)) \to ((X \vee Y) \to Z)$
10. $((X \to Y) \wedge (X \to (Y \to Z))) \to (X \to Z)$

Rule of inference:

- (Modus Ponens) $\dfrac{X,\ X \to Y}{Y}$

The rule of Modus Ponens means that if X and $X \to Y$ are derivable from the axiom system, then so is Y.

Deriving other equivalences from the above axioms.

Example 1. Derive $(A \wedge B) \rightarrow (B \wedge A)$ from the above axioms.

1. $(((A \wedge B) \rightarrow A) \wedge ((A \wedge B) \rightarrow (A \rightarrow (B \wedge A))) \rightarrow ((A \wedge B) \rightarrow (B \wedge A))$
   axiom (10), replace X with $(A \wedge B)$, Y with A, and Z with $(B \wedge A)$
2. $(A \wedge B) \rightarrow A$ — axiom (1)
3. $(((A \wedge B) \rightarrow B) \wedge ((A \wedge B) \rightarrow (B \rightarrow (A \rightarrow (B \wedge A))))) \rightarrow$ $((A \wedge B) \rightarrow (A \rightarrow (B \wedge A)))$
   axiom (10), replace X with $(A \wedge B)$, Y with B, and Z with $[A \rightarrow (B \wedge A)]$
4. $(A \wedge B) \rightarrow B$ — axiom (2)
5. $(B \rightarrow (A \rightarrow (B \wedge A))) \rightarrow ((A \wedge B) \rightarrow (B \rightarrow (A \rightarrow (B \wedge A))))$
   axiom (6), replace X with $(B \rightarrow (A \rightarrow (B \wedge A)))$ and Y with $(A \wedge B)$
6. $B \rightarrow (A \rightarrow (B \wedge A))$ — axiom (7)
7. $(A \wedge B) \rightarrow (B \rightarrow (A \rightarrow (B \wedge A)))$ — Modus Ponens, 6, 5
8. $((A \wedge B) \rightarrow B) \rightarrow (((A \wedge B) \rightarrow (B \rightarrow (A \rightarrow (B \wedge A)))) \rightarrow$ $(((A \wedge B) \rightarrow B) \wedge ((A \wedge B) \rightarrow (B \rightarrow (A \rightarrow (B \wedge A))))))$
   axiom (7), replace X with $((A \wedge B) \rightarrow B)$ and Y with $((A \wedge B) \rightarrow (B \rightarrow (A \rightarrow (B \wedge A))))$
9. $((A \wedge B) \rightarrow (B \rightarrow (A \rightarrow (B \wedge A)))) \rightarrow$ $(((A \wedge B) \rightarrow B) \wedge ((A \wedge B) \rightarrow (B \rightarrow (A \rightarrow (B \wedge A)))))$
   Modus Ponens, 4, 8
10. $((A \wedge B) \rightarrow B) \wedge ((A \wedge B) \rightarrow (B \rightarrow (A \rightarrow (B \wedge A))))$
   Modus Ponens, 7, 9
11. $(A \wedge B) \rightarrow (A \rightarrow (B \wedge A))$ — Modus Ponens, 3, 10
12. $((A \wedge B) \rightarrow A) \rightarrow (((A \wedge B) \rightarrow (A \rightarrow (B \wedge A))) \rightarrow$ $(((A \wedge B) \rightarrow A) \wedge ((A \wedge B) \rightarrow (A \rightarrow (B \wedge A)))))$
   axiom (7), replace X with $((A \wedge B) \rightarrow A)$ and Y with $((A \wedge B) \rightarrow (A \rightarrow (B \wedge A)))$
13. $((A \wedge B) \rightarrow (A \rightarrow (B \wedge A))) \rightarrow$ $(((A \wedge B) \rightarrow A) \wedge ((A \wedge B) \rightarrow (A \rightarrow (B \wedge A))))$ — Modus Ponens, 2, 12
14. $((A \wedge B) \rightarrow A) \wedge ((A \wedge B) \rightarrow (A \rightarrow (B \wedge A)))$ — Modus Ponens, 11, 13
15. $(A \wedge B) \rightarrow (B \wedge A)$ — Modus Ponens, 1, 14