## 2.8 Logical Equivalence

Figure 2.12 shows a truth table for the two statements $P \Rightarrow Q$ and $(\sim P) \vee Q$. The corresponding columns of these compound statements are identical; in other words, these two compound statements have exactly the same truth value for every combination of truth values of the statements $P$ and $Q$. In general, whenever two (compound) statements $R$ and $S$ have the same truth values for all combinations of truth values of their component statements, then we say that $R$ and $S$ are **logically equivalent** and indicate this by writing $R \equiv S$. Hence $P \Rightarrow Q$ and $(\sim P) \vee Q$ are logically equivalent and so $P \Rightarrow Q \equiv (\sim P) \vee Q$.

Another, even simpler, example of logical equivalence concerns $P \wedge Q$ and $Q \wedge P$. That $P \wedge Q \equiv Q \wedge P$ is verified in the truth table shown in Figure 2.13.

What is the practical significance of logical equivalence? Suppose that $R$ and $S$ are logically equivalent compound statements. Then we know that $R$ and $S$ have the same truth values for all possible combinations of truth values of their component statements. But this means that the biconditional $R \Leftrightarrow S$ is true for all possible combinations of truth values of their component statements and hence $R \Leftrightarrow S$ is a tautology. Conversely, if $R \Leftrightarrow S$ is a tautology, then $R$ and $S$ are logically equivalent.

| $P$ | $Q$ | $\sim P$ | $P \Rightarrow Q$ | $(\sim P) \vee Q$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $\boldsymbol{T}$ | $\boldsymbol{T}$ |
| $T$ | $F$ | $F$ | $\boldsymbol{F}$ | $\boldsymbol{F}$ |
| $F$ | $T$ | $T$ | $\boldsymbol{T}$ | $\boldsymbol{T}$ |
| $F$ | $F$ | $T$ | $\boldsymbol{T}$ | $\boldsymbol{T}$ |

**Figure 2.12** Verification of $P \Rightarrow Q \equiv (\sim P) \vee Q$

| $P$ | $Q$ | $P \wedge Q$ | $Q \wedge P$ |
|---|---|---|---|
| $T$ | $T$ | $\boldsymbol{T}$ | $\boldsymbol{T}$ |
| $T$ | $F$ | $\boldsymbol{F}$ | $\boldsymbol{F}$ |
| $F$ | $T$ | $\boldsymbol{F}$ | $\boldsymbol{F}$ |
| $F$ | $F$ | $\boldsymbol{F}$ | $\boldsymbol{F}$ |

**Figure 2.13** Verification of $P \wedge Q \equiv Q \wedge P$

Let $R$ be a mathematical statement that we would like to show is true, and suppose that $R$ and some statement $S$ are logically equivalent. If we can show that $S$ is true, then $R$ is true as well. For example, suppose that we want to verify the truth of an implication $P \Rightarrow Q$. If we can establish the truth of the statement $(\sim P) \vee Q$, then the logical equivalence of $P \Rightarrow Q$ and $(\sim P) \vee Q$ guarantees that $P \Rightarrow Q$ is true as well.

**Example 2.16** *Returning to the mathematics instructor in Example 2.6 and whether she kept her promise that*

*If you earn an A on the final exam, then you will receive an A for the final grade.*

*we need know only that the student did not receive an A on the final exam or the student received an A as a final grade to see that she kept her promise.* ♦

Since the logical equivalence of $P \Rightarrow Q$ and $(\sim P) \vee Q$, verified in Figure 2.12, is especially important and we will have occasion to use this fact often, we state it as a theorem.

**Theorem 2.17** *Let $P$ and $Q$ be two statements. Then*

$$P \Rightarrow Q \text{ and } (\sim P) \vee Q$$

*are logically equivalent.*

Let's return to the truth table in Figure 2.13, where we showed that $P \wedge Q$ and $Q \wedge P$ are logically equivalent for any two statements $P$ and $Q$. In particular, this says that

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P) \text{ and } (Q \Rightarrow P) \wedge (P \Rightarrow Q)$$

are logically equivalent. Of course, $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is precisely what is called the biconditional of $P$ and $Q$. Since $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ and $(Q \Rightarrow P) \wedge (P \Rightarrow Q)$ are logically equivalent, $(Q \Rightarrow P) \wedge (P \Rightarrow Q)$ represents the biconditional of $P$ and $Q$ as well. Since $Q \Rightarrow P$ can be written as "$P$ if $Q$" and $P \Rightarrow Q$ can be expressed as "$P$ only if $Q$", their conjunction can be written as "$P$ if $Q$ and $P$ only if $Q$" or, more simply, as

$$P \text{ if and only if } Q.$$

Consequently, expressing $P \Leftrightarrow Q$ as "$P$ if and only if $Q$" is justified. Furthermore, since $Q \Rightarrow P$ can be phrased as "$P$ is necessary for $Q$" and $P \Rightarrow Q$ can be expressed as "$P$ is sufficient for $Q$", writing $P \Leftrightarrow Q$ as "$P$ is necessary and sufficient for $Q$" is likewise justified.

## 2.9 Some Fundamental Properties of Logical Equivalence

It probably comes as no surprise that the statements $P$ and $\sim(\sim P)$ are logically equivalent. This fact is verified in Figure 2.14.

| $P$ | $\sim P$ | $\sim(\sim P)$ |
|---|---|---|
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $F$ |

**Figure 2.14** Verification of $P \equiv \sim(\sim P)$

We mentioned in Figure 2.13 that, for two statements $P$ and $Q$, the statements $P \wedge Q$ and $Q \wedge P$ are logically equivalent. There are other fundamental logical equivalences that we often encounter as well.

**Theorem 2.18** *For statements $P$, $Q$, and $R$,*

(1) *Commutative Laws*
  (a) $P \vee Q \equiv Q \vee P$
  (b) $P \wedge Q \equiv Q \wedge P$
(2) *Associative Laws*
  (a) $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
  (b) $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
(3) *Distributive Laws*
  (a) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
  (b) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
(4) *De Morgan's Laws*
  (a) $\sim(P \vee Q) \equiv (\sim P) \wedge (\sim Q)$
  (b) $\sim(P \wedge Q) \equiv (\sim P) \vee (\sim Q)$.

Each part of Theorem 2.18 is verified by means of a truth table. We have already established the commutative law for conjunction (namely, that $P \wedge Q \equiv Q \wedge P$) in Figure 2.13. In Figure 2.15 $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ is verified by observing that the columns corresponding to the statements $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$ are identical.

The laws given in Theorem 2.18, together with other known logical equivalences, can be used to good advantage at times to prove other logical equivalences (without introducing a truth table).

| $P$ | $Q$ | $R$ | $Q \wedge R$ | $P \vee (Q \wedge R)$ | $P \vee Q$ | $P \vee R$ | $(P \vee Q) \wedge (P \vee R)$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | **$T$** | $T$ | $T$ | **$T$** |
| $T$ | $T$ | $F$ | $F$ | **$T$** | $T$ | $T$ | **$T$** |
| $T$ | $F$ | $T$ | $F$ | **$T$** | $T$ | $T$ | **$T$** |
| $T$ | $F$ | $F$ | $F$ | **$T$** | $T$ | $T$ | **$T$** |
| $F$ | $T$ | $T$ | $T$ | **$T$** | $T$ | $T$ | **$T$** |
| $F$ | $T$ | $F$ | $F$ | **$F$** | $T$ | $F$ | **$F$** |
| $F$ | $F$ | $T$ | $F$ | **$F$** | $F$ | $T$ | **$F$** |
| $F$ | $F$ | $F$ | $F$ | **$F$** | $F$ | $F$ | **$F$** |

**Figure 2.15** Verification of the distributive law $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

**Example 2.19** *Suppose that we are asked to prove that*

$$\sim(P \Rightarrow Q) \equiv P \wedge (\sim Q)$$

*for every two statements P and Q. Using the logical equivalence of $P \Rightarrow Q$ and $(\sim P) \vee Q$ from Theorem 2.17 and Theorem 2.18(4a), we have the following:*

$$\sim(P \Rightarrow Q) \equiv \sim((\sim P) \vee Q) \equiv (\sim(\sim P)) \wedge (\sim Q) \equiv P \wedge (\sim Q), \qquad \textbf{(2.1)}$$

*implying that the statements $\sim(P \Rightarrow Q)$ and $P \wedge (\sim Q)$ are logically equivalent, which we alluded to earlier.* ♦

It is important to keep in mind what we have said about logical equivalence. For example, the logical equivalence of $P \wedge Q$ and $Q \wedge P$ allows us to replace a statement of the type $P \wedge Q$ by $Q \wedge P$ without changing its truth value. As an additional example, according to De Morgan's Laws in Theorem 2.18, if it is not the case that an integer $a$ is even or an integer $b$ is even, then it follows that $a$ and $b$ are both odd.

**Example 2.20** *Using the second of De Morgan's Laws and (2.1), we can establish a useful logically equivalent form of the negation of $P \Leftrightarrow Q$ by the following string of logical equivalences:*

$$\begin{aligned} \sim(P \Leftrightarrow Q) &\equiv \sim((P \Rightarrow Q) \wedge (Q \Rightarrow P)) \\ &\equiv (\sim(P \Rightarrow Q)) \vee (\sim(Q \Rightarrow P)) \\ &\equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P)). \end{aligned}$$

♦

What we have observed about the negation of an implication and a biconditional is repeated in the following theorem.

**Theorem 2.21** *For statements P and Q,*

(a) $\sim(P \Rightarrow Q) \equiv P \wedge (\sim Q)$
(b) $\sim(P \Leftrightarrow Q) \equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$.