2.10 Quantified Statements

We have mentioned that if $P(x)$ is an open sentence over a domain S, then $P(x)$ is a statement for each $x \in S$. We illustrate this again.

Example 2.22 *If* $S = \{1, 2, \cdots, 7\}$, *then*

$$P(n): \frac{2n^2 + 5 + (-1)^n}{2} \text{ is prime.}$$

is a statement for each $n \in S$. *Therefore,*

$$\begin{aligned} P(1)&: 3 \text{ is prime.} \\ P(2)&: 7 \text{ is prime.} \\ P(3)&: 11 \text{ is prime.} \\ P(4)&: 19 \text{ is prime.} \end{aligned}$$

are true statements, while

$$P(5): \ 27 \textit{ is prime.}$$
$$P(6): \ 39 \textit{ is prime.}$$
$$P(7): \ 51 \textit{ is prime.}$$

are false statements. ♦

There are other ways that an open sentence can be converted into a statement, namely by a method called **quantification**. Let $P(x)$ be an open sentence over a domain S. Adding the phrase "For every $x \in S$" to $P(x)$ produces a statement called a **quantified statement**. The phrase "for every" is referred to as the **universal quantifier** and is denoted by the symbol $\forall$. Other ways to express the universal quantifier are "for each" and "for all". This quantified statement is expressed in symbols by

$$\forall x \in S, P(x) \tag{2.2}$$

and is expressed in words by

$$\text{For every } x \in S, P(x). \tag{2.3}$$

The quantified statement (2.2) (or (2.3)) is true if $P(x)$ is true for every $x \in S$; while the quantified statement (2.2) is false if $P(x)$ is false for at least one element $x \in S$.

Another way to convert an open sentence $P(x)$ over a domain S into a statement through quantification is by the introduction of a quantifier called an existential quantifier. Each of the phrases "there exists", "there is", "for some", and "for at least one" is referred to as an **existential quantifier** and is denoted by the symbol $\exists$. The quantified statement

$$\exists x \in S, P(x) \tag{2.4}$$

can be expressed in words by

$$\text{There exists } x \in S \text{ such that } P(x). \tag{2.5}$$

The quantified statement (2.4) (or (2.5)) is true if $P(x)$ is true for at least one element $x \in S$, while the quantified statement (2.4) is false if $P(x)$ is false for all $x \in S$.

We now consider two quantified statements constructed from the open sentence we saw in Example 2.22.

Example 2.23 *For the open sentence*

$$P(n): \ \frac{2n^2+5+(-1)^n}{2} \textit{ is prime.}$$

over the domain $S = \{1, 2, \cdots, 7\}$, *the quantified statement*

$$\forall n \in S, P(n): \textit{ For every } n \in S, \ \frac{2n^2+5+(-1)^n}{2} \textit{ is prime.}$$

is false since $P(5)$ *is false, for example; while the quantified statement*

$$\exists n \in S, P(n): \textit{ There exists } n \in S \textit{ such that } \frac{2n^2+5+(-1)^n}{2} \textit{ is prime.}$$

is true since $P(1)$ *is true, for example.* ♦

The quantified statement $\forall x \in S, P(x)$ can also be expressed as

$$\text{If } x \in S, \text{ then } P(x).$$

Consider the open sentence $P(x): x^2 \geq 0.$ over the set $\mathbf{R}$ of real numbers. Then

$$\forall x \in \mathbf{R}, P(x)$$

or, equivalently,

$$\forall x \in \mathbf{R}, x^2 \geq 0$$

can be expressed as

For every real number x, $x^2 \geq 0$.

or

If x is a real number, then $x^2 \geq 0$.

as well as

The square of every real number is nonnegative.

In general, the universal quantifier is used to claim that the statement resulting from a given open sentence is true when each value of the domain of the variable is assigned to the variable. Consequently, the statement $\forall x \in \mathbf{R}, x^2 \geq 0$ is true since $x^2 \geq 0$ is true for every real number x.

Suppose now that we were to consider the open sentence $Q(x): x^2 \leq 0$. The statement $\forall x \in \mathbf{R}, Q(x)$ (that is, for every real number x, we have $x^2 \leq 0$) is false since, for example, $Q(1)$ is false. Of course, this means that its negation is true. If it were not the case that for every real number x, we have $x^2 \leq 0$, then there must exist some real number x such that $x^2 > 0$. This negation

There exists a real number x such that $x^2 > 0$.

can be written in symbols as

$$\exists x \in \mathbf{R}, x^2 > 0 \quad \text{or} \quad \exists x \in \mathbf{R}, \sim Q(x).$$

More generally, if we are considering an open sentence $P(x)$ over a domain S, then

$$\sim(\forall x \in S, P(x)) \equiv \exists x \in S, \sim P(x).$$

Example 2.24 *Suppose that we are considering the set $A = \{1, 2, 3\}$ and its power set $\mathcal{P}(A)$, the set of all subsets of A. Then the quantified statement*

$$\textit{For every set } B \in \mathcal{P}(A),\ A - B \neq \emptyset. \qquad \mathbf{(2.6)}$$

is false since for the subset $B = A = \{1, 2, 3\}$, we have $A - B = \emptyset$. The negation of the statement (2.6) is

$$\textit{There exists } B \in \mathcal{P}(A) \textit{ such that } A - B = \emptyset. \qquad \mathbf{(2.7)}$$

The statement (2.7) is therefore true since for $B = A \in \mathcal{P}(A)$, we have $A - B = \emptyset$. The statement (2.6) can also be written as

$$\textit{If } B \subseteq A, \textit{ then } A - B \neq \emptyset. \qquad \mathbf{(2.8)}$$

Consequently, the negation of (2.8) can be expressed as

$$\textit{There exists some subset } B \textit{ of } A \textit{ such that } A - B = \emptyset. \quad \blacklozenge$$

The existential quantifier is used to claim that at least one statement resulting from a given open sentence is true when the values of a variable are assigned from its domain. We know that for an open sentence $P(x)$ over a domain S, the quantified statement $\exists x \in S,\ P(x)$ is true provided $P(x)$ is a true statement for at least one element $x \in S$. Thus the statement $\exists x \in \mathbf{R},\ x^2 > 0$ is true since, for example, $x^2 > 0$ is true for $x = 1$.

The quantified statement

$$\exists x \in \mathbf{R},\ 3x = 12$$

is therefore true since there is some real number x for which $3x = 12$, namely $x = 4$ has this property. (Indeed, $x = 4$ is the *only* real number for which $3x = 12$.) On the other hand, the quantified statement

$$\exists n \in \mathbf{Z},\ 4n - 1 = 0$$

is false as there is no integer n for which $4n - 1 = 0$. (Of course, $4n - 1 = 0$ when $n = 1/4$ but $1/4$ is not an integer.)

Suppose that $Q(x)$ is an open sentence over a domain S. If the statement $\exists x \in S,\ Q(x)$ is *not* true, then it must be the case that for every $x \in S$, $Q(x)$ is false. That is,

$$\sim(\exists x \in S,\ Q(x)) \equiv \forall x \in S,\ \sim Q(x).$$

We illustrate this with a specific example.

Example 2.25 *The following statement contains the existential quantifier:*

$$\textit{There exists a real number } x \textit{ such that } x^2 = 3. \quad \textbf{(2.9)}$$

If we let $P(x) : x^2 = 3$, then (2.9) can be rewritten as $\exists x \in \mathbf{R},\ P(x)$. The statement (2.9) is true since $P(x)$ is true when $x = \sqrt{3}$ (or when $x = -\sqrt{3}$). Hence the negation of (2.9) is:

$$\textit{For every real number } x,\ x^2 \neq 3. \quad \textbf{(2.10)}$$

The statement (2.10) is therefore false. ♦

Let $P(x, y)$ be an open sentence, where the domain of the variable x is S and the domain of the variable y is T. Then the quantified statement

$$\text{For all } x \in S \text{ and } y \in T,\ P(x, y).$$

can be expressed symbolically as

$$\forall x \in S, \forall y \in T,\ \ P(x, y). \quad \textbf{(2.11)}$$

The negation of the statement (2.11) is

$$\begin{aligned} \sim(\forall x \in S, \forall y \in T,\ P(x, y)) &\equiv \exists x \in S,\ \sim(\forall y \in T,\ P(x, y)) \\ &\equiv \exists x \in S, \exists y \in T,\ \sim P(x, y). \quad \textbf{(2.12)} \end{aligned}$$

We now consider examples of quantified statements involving two variables.

Example 2.26 *Consider the statement*

$$\text{For every two real numbers } x \text{ and } y,\ x^2 + y^2 \geq 0. \tag{2.13}$$

If we let

$$P(x, y):\ x^2 + y^2 \geq 0$$

where the domain of both x and y is $\mathbf{R}$, then statement (2.13) can be expressed as

$$\forall x \in \mathbf{R}, \forall y \in \mathbf{R},\ P(x, y) \tag{2.14}$$

or as

$$\forall x, y \in \mathbf{R},\ P(x, y).$$

Since $x^2 \geq 0$ and $y^2 \geq 0$ for all real numbers x and y and so $x^2 + y^2 \geq 0$, $P(x, y)$ is true for all real numbers x and y and the quantified statement (2.14) is true.

The negation of statement (2.14) is therefore

$$\sim(\forall x \in \mathbf{R}, \forall y \in \mathbf{R},\ P(x, y)) \equiv \exists x \in \mathbf{R}, \exists y \in \mathbf{R},\ \sim P(x, y), \tag{2.15}$$

which, in words, is

$$\text{There exist real numbers } x \text{ and } y \text{ such that } x^2 + y^2 < 0. \tag{2.16}$$

The statement (2.16) is therefore false. ♦

For an open sentence containing two variables, the domains of the variables need not be the same.

Example 2.27 *Consider the statement*

$$\text{For every } s \in S \text{ and } t \in T,\ st + 2 \text{ is a prime.} \tag{2.17}$$

where the domain of the variable s is $S = \{1, 3, 5\}$ and the domain of the variable t is $T = \{3, 9\}$. If we let

$$Q(s, t):\ st + 2 \text{ is a prime.}$$

then the statement (2.17) can be expressed as

$$\forall s \in S, \forall t \in T,\ Q(s, t). \tag{2.18}$$

Since all of the statements

$Q(1, 3):\ 1 \cdot 3 + 2$ *is a prime.* $Q(3, 3):\ 3 \cdot 3 + 2$ *is a prime.*
$Q(5, 3):\ 5 \cdot 3 + 2$ *is a prime.*

$Q(1, 9):\ 1 \cdot 9 + 2$ *is a prime.* $Q(3, 9):\ 3 \cdot 9 + 2$ *is a prime.*
$Q(5, 9):\ 5 \cdot 9 + 2$ *is a prime.*

are true, the quantified statement (2.18) is true.

As we saw in (2.12), the negation of the quantified statement (2.18) is

$$\sim(\forall s \in S, \forall t \in T,\ Q(s,t)) \equiv \exists s \in S, \exists t \in T,\ \sim Q(s,t)$$

and so the negation of (2.17) is

There exist $s \in S$ and $t \in T$ such that $st + 2$ is not a prime. **(2.19)**

The statement (2.19) is therefore false. ♦

Again, let $P(x, y)$ be an open sentence, where the domain of the variable x is S and the domain of the variable y is T. The quantified statement

There exist $x \in S$ and $y \in T$ such that $P(x, y)$.

can be expressed in symbols as

$$\exists x \in S, \exists y \in T,\ P(x, y). \tag{2.20}$$

The negation of the statement (2.20) is

$$\begin{aligned}\sim(\exists x \in S, \exists y \in T,\ P(x, y)) &\equiv \forall x \in S,\ \sim(\exists y \in T,\ P(x, y)) \\ &\equiv \forall x \in S, \forall y \in T,\ \sim P(x, y).\end{aligned} \tag{2.21}$$

We now illustrate this situation.

Example 2.28 *Consider the open sentence*

$$R(s,t):\ |s-1| + |t-2| \le 2,$$

where the domain of the variable s is the set S of even integers and the domain of the variable t is the set T of odd integers. Then the quantified statement

$$\exists s \in S, \exists t \in T, R(s,t) \tag{2.22}$$

can be expressed in words as

There exist an even integer s and an odd integer t such that $|s-1| + |t-2| \le 2$. **(2.23)**

Since $R(2, 3):\ 1 + 1 \le 2$ is true, the quantified statement (2.23) is true.

The negation of (2.22) is therefore

$$\sim(\exists s \in S, \exists t \in T,\ R(s,t)) \equiv \forall s \in S, \forall t \in T,\ \sim R(s,t) \tag{2.24}$$

and so the negation of (2.22), in words, is

For every even integer s and every odd integer t, $|s-1| + |t-2| > 2$. **(2.25)**

The quantified statement (2.25) is therefore false. ♦

Quantified statements may contain both universal and existential quantifiers. We will encounter this in Section 7.2.