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Logic

In mathematics our goal is to seek the truth. Are there connections between two given mathematical concepts? If so, what are they? Under what conditions does an object possess a particular property? Finding answers to questions such as these is important, but we cannot be satisfied only with this. We must be certain that we are right and that our explanation for why we believe we are correct is convincing to others. The reasoning we use as we proceed from what we know to what we wish to show must be logical. It must make sense to others, not just to ourselves.

There is joint responsibility here, however. It is the writer's responsibility to use the rules of logic to give a valid and clear argument with enough details provided to allow the reader to understand what we have written and to be convinced. It is the reader's responsibility to know the basics of logic and to study the concepts involved sufficiently well so that he or she will not only be able to understand a well-presented argument but can decide as well whether it is valid. Consequently, both writer and reader must have some familiarity with logic.

Although it is possible to spend a great deal of time studying logic, we will present only what we actually need and will instead use the majority of our time putting what we learn into practice.

2.1 Statements

In mathematics we are constantly dealing with statements. By a **statement** we mean a declarative sentence or assertion that is true or false (but not both). Statements therefore declare or assert the truth of something. Of course, the statements in which we will be primarily interested deal with mathematics. For example, the sentences

The integer 3 is odd.
The integer 57 is prime.

are statements (only the first of which is true).

Every statement has a **truth value**, namely **true** (denoted by T) or **false** (denoted by F). We often use P, Q, and R to denote statements, or perhaps $P_1, P_2, \ldots, P_n$ if several statements are involved. We have seen that

$$P_1 : \text{The integer 3 is odd.}$$

and

$$P_2 : \text{The integer 57 is prime.}$$

are statements, where P_1 has truth value T and P_2 has truth value F.

Sentences that are imperative (commands) such as

Substitute the number 2 for x.
Find the derivative of $f(x) = e^{-x}\cos 2x$.

or are interrogative (questions) such as

Are these sets disjoint?
What is the derivative of $f(x) = e^{-x}\cos 2x$?

or are exclamatory such as

What an interesting question!
How difficult this problem is!

are not statements since these sentences are not declarative.

It may not be immediately clear whether a statement is true or false. For example, the sentence "The 100th digit in the decimal expansion of π is 7." is a statement, but it may be necessary to check some table to determine whether this statement is true. Indeed, for a sentence to be a statement, it is not a requirement that we be able to determine its truth value.

The sentence "The real number r is rational." is a statement *provided* we know what real number r is being referred to. Without this additional information, however, it is impossible to assign a truth value to it. This is an example of what is often referred to as an open sentence. In general, an **open sentence** is a declarative sentence that contains one or more variables, each variable representing a value in some prescribed set, called the **domain** of the variable, and which becomes a statement when values from their respective domains are substituted for these variables. For example, the open sentence "$3x = 12$" where the domain of x is the set of integers is a true statement only when $x = 4$.

An open sentence that contains a variable x is typically represented by $P(x)$, $Q(x)$, or $R(x)$. If $P(x)$ is an open sentence, where the domain of x is S, then we say $P(x)$ is an **open sentence over the domain** S. Also, $P(x)$ is a statement for each $x \in S$. For example, the open sentence

$$P(x):\ (x-3)^2 \le 1$$

over the domain $\mathbf{Z}$ is a true statement when $x \in \{2, 3, 4\}$ and is a false statement otherwise.

Example 2.1 *For the open sentence*

$$P(x, y):\ |x+1| + |y| = 1$$

P
T
F

Q
T
F

P	Q
T	T
T	F
F	T
F	F

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Figure 2.1 Truth tables for one, two, and three statements

in two variables, suppose that the domain of the variable x *is* $S = \{-2, -1, 0, 1\}$ *and the domain of the variable* y *is* $T = \{-1, 0, 1\}$. *Then*

$$P(-1, 1) : \ |(-1) + 1| + |1| = 1$$

is a true statement, while

$$P(1, -1) : \ |1 + 1| + |-1| = 1$$

is a false statement. In fact, $P(x, y)$ *is a true statement when*

$$(x, y) \in \{(-2, 0), (-1, -1), (-1, 1), (0, 0)\},$$

while $P(x, y)$ *is a false statement for all other elements* $(x, y) \in S \times T$. ♦

The possible truth values of a statement are often given in a table, called a **truth table.** The truth tables for two statements P and Q are given in Figure 2.1. Since there are two possible truth values for each of P and Q, there are four possible combinations of truth values for P and Q. The truth table showing all these combinations is also given in Figure 2.1. If a third statement R is involved, then there are eight possible combinations of truth values for P, Q, and R. This is displayed in Figure 2.1 as well. In general, a truth table involving n statements $P_1, P_2, \cdots, P_n$ contains 2^n possible combinations of truth values for these statements, and a truth table showing these combinations would have n columns and 2^n rows. Much of the time, we will be dealing with two statements, usually denoted by P and Q; so the associated truth table will have four rows with the first two columns headed by P and Q. In this case, it is customary to consider the four combinations of the truth values in the order TT, TF, FT, FF, from top to bottom.

2.2 The Negation of a Statement

Much of the interest in integers and other familiar sets of numbers comes not only from the numbers themselves but from properties of the numbers that result by performing operations on them (such as taking their negatives, adding or multiplying them, or combinations of these). Similarly, much of our interest in statements comes from investigating the truth or falseness of new statements that can be produced from one or more given statements by performing certain operations on them. Our first example concerns producing a new statement from a single given statement.

The **negation** of a statement P is the statement:

not P.

and is denoted by $\sim P$. Although $\sim P$ could always be expressed as

It is not the case that P**.**

there are usually better ways to express the statement $\sim P$.

Example 2.2 *For the statement*

P_1 : *The integer* 3 *is odd.*

described above, we have

$\sim P_1$: *It is not the case that the integer* 3 *is odd.*

but it would be much preferred to write

$\sim P_1$: *The integer* 3 *is not odd.*

or better yet to write

$\sim P_1$: *The integer* 3 *is even.*

Similarly, the negation of the statement

P_2 : *The integer* 57 *is prime.*

considered above is

$\sim P_2$: *The integer* 57 *is not prime.*

Note that $\sim P_1$ *is false, while* $\sim P_2$ *is true.* ♦

Indeed, the negation of a true statement is always false, and the negation of a false statement is always true; that is, the truth value of $\sim P$ is opposite to that of P. This is summarized in Figure 2.2, which gives the truth table for $\sim P$ (in terms of the possible truth values of P).

P	$\sim P$
T	F
F	T

Figure 2.2 The truth table for negation

2.3 The Disjunction and Conjunction of Statements

For two given statements P and Q, a common way to produce a new statement from them is by inserting the word "or" or "and" between P and Q. The **disjunction** of the statements P and Q is the statement:

$$P \textbf{ or } Q.$$

and is denoted by $P \vee Q$. The disjunction $P \vee Q$ is true if at least one of P and Q is true; otherwise, $P \vee Q$ is false. Therefore, $P \vee Q$ is true if exactly one of P and Q is true or if both P and Q are true.

Example 2.3 *For the statements*

$$P_1 : \textit{The integer } 3 \textit{ is odd. and } P_2 : \textit{The integer } 57 \textit{ is prime.}$$

described earlier, the disjunction is the new statement

$$P_1 \vee P_2\textit{: Either } 3 \textit{ is odd or } 57 \textit{ is prime.}$$

which is true since at least one of P_1 and P_2 is true (namely, P_1 is true). Of course, in this case exactly one of P_1 and P_2 is true. ♦

For two statements P and Q, the truth table for $P \vee Q$ is shown in Figure 2.3. This truth table then describes precisely when $P \vee Q$ is true (or false).

Although the truth of "P or Q" allows for both P and Q to be true, there are instances when the use of "or" does not allow that possibility. For example, for an integer n, if we were to say "n is even or n is odd", then surely it is not possible for both "n is even" and "n is odd" to be true. When "or" is used in this manner, it is called the **exclusive or**. Suppose, for example, that $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$, where $k \geq 2$, is a partition of a set S and x is some element of S. If

$$x \in S_1 \text{ or } x \in S_2$$

is true, then it is impossible for both $x \in S_1$ and $x \in S_2$ to be true.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Figure 2.3 The truth table for disjunction

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Figure 2.4 The truth table for conjunction

The **conjunction** of the statements P and Q is the statement:

P and Q.

and is denoted by $P \wedge Q$. The conjunction $P \wedge Q$ is true only when both P and Q are true; otherwise, $P \wedge Q$ is false.

Example 2.4 *For P_1 : The integer 3 is odd. and P_2 : The integer 57 is prime, the statement*

$$P_1 \wedge P_2 : \text{3 is odd and 57 is prime.}$$

is false since P_2 is false and so not both P_1 and P_2 are true. ♦

The truth table for the conjunction of two statements is shown in Figure 2.4.

2.4 The Implication

The statement formed from two given statements in which we will be most interested is the implication (also called the conditional). For statements P and Q, the **implication** (or **conditional**) is the statement:

If P, then Q.

and is denoted by $P \Rightarrow Q$. In addition to the wording "If P, then Q.", we also express $P \Rightarrow Q$ in words as

P implies Q.

The truth table for $P \Rightarrow Q$ is given in Figure 2.5.

Notice that $P \Rightarrow Q$ is false when P is true and Q is false, and is true otherwise.

Example 2.5 *For P_1 : The integer 3 is odd. and P_2 : The integer 57 is prime, the implication*

$$P_1 \Rightarrow P_2 : \text{If 3 is an odd integer, then 57 is prime.}$$

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Figure 2.5 The truth table for implication

is a false statement. The implication

$$P_2 \Rightarrow P_1 : \textit{If } 57 \textit{ is prime, then } 3 \textit{ is odd.}$$

is true, however. ♦

While the truth tables for the negation $\sim P$, the disjunction $P \vee Q$, and the conjunction $P \wedge Q$ are probably not unexpected, this may not be so for the implication $P \Rightarrow Q$. There is ample justification, however, for the truth values in the truth table of $P \Rightarrow Q$. We illustrate this with an example.

Example 2.6 *A student is taking a math class (let's say this one) and is currently receiving a B+. He visits his instructor a few days before the final examination and asks her, "Is there any chance that I can get an A in this course?" His instructor looks through her grade book and says, "If you earn an A on the final exam, then you will receive an A for your final grade." We now check the truth or falseness of this implication based on the various combinations of truth values of the statements*

$$P : \textit{You earn an A on the final exam.}$$

and

$$Q : \textit{You receive an A for your final grade.}$$

which make up the implication.

Analysis Suppose first that P and Q are both true. That is, the student receives an A on his final exam and later learns that he got an A for his final grade in the course. Did his instructor tell the truth? I think we would all agree that she did. So if P and Q are both true, then so too is $P \Rightarrow Q$, which agrees with the first row of the truth table of Figure 2.5.

Second, suppose that P is true and Q is false. So the student got an A on his final exam but did not receive an A as a final grade, say he received a B. Certainly, his instructor did not do as she promised (as she will soon be reminded by her student). What she said was false, which agrees with the second row of the table in Figure 2.5.

Third, suppose that P is false and Q is true. In this case, the student did not get an A on his final exam (say he earned a B), *but* when he received his final grades, he learned (and was pleasantly surprised) that his final grade was an A. How could this happen? Perhaps his instructor was lenient. Perhaps the final exam was unusually difficult, and a grade of B on it indicated an exceptionally good performance. Perhaps the instructor made a mistake. In any case, the instructor did not lie; so she told the truth. This agrees with the third row of the table in Figure 2.5.

Finally, suppose that P and Q are both false. That is, suppose the student did not get an A on his final exam, and he also did not get an A for a final grade. The instructor did not lie here either. She only promised the student an A *if* he got an A on the final exam. She promised nothing if the student did not get an A on the final exam. So the instructor told the truth, and this agrees with the fourth and final row of the table. ♦

2.6 The Biconditional

For statements (or open sentences) P and Q, the implication $Q \Rightarrow P$ is called the **converse** of $P \Rightarrow Q$. The converse of an implication will often be of interest to us, either by itself or in conjunction with the original implication.

Example 2.11 *For the statements*

$$P_1 : \text{ 3 is an odd integer.} \qquad P_2 : \text{ 57 is prime.}$$

the converse of the implication

$$P_1 \Rightarrow P_2 : \textit{ If 3 is an odd integer, then 57 is prime.}$$

is the implication

$$P_2 \Rightarrow P_1 : \textit{ If 57 is prime, then 3 is an odd integer.}$$ ♦

For statements (or open sentences) P and Q, the conjunction

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

of the implication $P \Rightarrow Q$ and its converse is called the **biconditional** of P and Q and is denoted by $P \Leftrightarrow Q$. For statements P and Q, the truth table for $P \Leftrightarrow Q$ can therefore be determined. This is given in Figure 2.7. From this table, we see that $P \Leftrightarrow Q$ is true whenever the statements P and Q are both true or are both false, while $P \Leftrightarrow Q$ is false otherwise. That is, $P \Leftrightarrow Q$ is true precisely when P and Q have the same truth values.

The biconditional $P \Leftrightarrow Q$ is often stated as

P is equivalent to Q.

or

P if and only if Q.

or as

P is a necessary and sufficient condition for Q.

For statements P and Q, it then follows that the biconditional "P if and only if Q" is true only when P and Q have the same truth values.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Figure 2.7 The truth table for a biconditional