Exclusive or.
Compound statements. Order of operations.
Logically equivalent statements.
Expressing some operations in terms of others.
Disjunctive and conjunctive normal forms.
Nand operation. Observations and questions.

Exclusive or (exclusive disjunction).

Exclusive or (aka exclusive disjunction) is an operation denoted by $\oplus$ and defined by the following truth table:

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

So, $P \oplus Q$ means "either P or Q, but not both."

Compound statements.

A compound statement is an expression with one or more propositional variable(s) and two or more logical connectives (operators).

Examples:

- $(P \wedge Q) \rightarrow R$
- $(\neg P \rightarrow (Q \vee P)) \vee (\neg Q)$
- $\neg(\neg P)$.

Given the truth value of each propositional variable, we can determine the truth value of the compound statement.

Order of operations.

Just like in algebra, expressions in parentheses are evaluated first. Otherwise, negations are performed first, then conjunctions, followed by disjunctions (including exclusive disjunctions), implications, and, finally, the biconditionals.

For example, if P is True, Q is False, and R is True, then the truth value of $\neg P \to Q \wedge (R \vee P)$ is computed as follows: first, in parentheses we have $R \vee P$ is True, next, $\neg P$ is False, then $Q \wedge (R \vee P)$ is False, so $\neg P \to Q \wedge (R \vee P)$ is True.

In other words, $\neg P \to Q \wedge (R \vee P)$ is equivalent to $(\neg P) \to (Q \wedge (R \vee P))$.

Remark. The order given above is the most often used one. Some authors use other orders. In particular, many authors treat $\wedge$ and $\vee$ equally (so these are evaluated in the order in which they appear in the expression, just like $\times$ and $\div$ in algebra), and $\to$ and $\leftrightarrow$ equally (but after $\wedge$ and $\vee$, like $+$ and $-$ are performed after $\times$ and $\div$). To avoid confusion in this class, we will use parentheses to make the order clear.

Logically equivalent statements.

Two compound statements are called logically equivalent if they always have the same truth values (for any combination of the truth values of their components, i.e. variables). In other words, they are logically equivalent when they have identical truth tables.

Example: $P \to Q$ and $\neg P \vee Q$ are logically equivalent. Indeed,

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Comparing the third and fifth columns of the above table, we see that the truth values of $P \to Q$ and $\neg P \vee Q$ are always the same.

Logical equivalence is denoted by the symbol $\equiv$, e.g. we write

$$P \to Q \equiv \neg P \vee Q.$$

Expressing some operations in terms of others.

From the six operations $\neg$, $\wedge$, $\vee$, $\oplus$, $\rightarrow$, $\leftrightarrow$, some operations can be expressed in terms of others. For example, as we saw above,

$P \rightarrow Q \equiv \neg P \vee Q.$

Also, it can be checked using the truth tables that

$P \wedge Q \equiv \neg(\neg P \vee \neg Q),$

$P \vee Q \equiv \neg(\neg P \wedge \neg Q),$

$P \oplus Q \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q),$

$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q).$

Disjunctive and conjunctive normal forms.

A formula is said to be in disjunctive normal form if it is a disjunction of one or more conjunctions of one or more propositional variable(s) and/or its/their negation(s).

More simply speaking, if the formula either consists of one "piece" or is a disjunction of two or more "pieces" where each "piece" is either a propositional variable, or its negation, or a conjunction of variables/negations of variables.

Examples of formulas in disjunctive normal form: $P \vee Q$, $P \wedge \neg Q$, $P \vee (Q \wedge \neg P)$, $(P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q) \vee (\neg Q \wedge R)$.

A formula is said to be in conjunctive normal form if it is a conjunction of one or more disjunctions of one or more propositional variable(s) and/or its/their negation(s).

Examples of formulas in conjunctive normal form: $P \wedge Q$, $P \vee \neg Q$, $P \wedge (Q \vee \neg P)$, $(P \vee Q \vee \neg R) \wedge (\neg P \vee Q) \wedge (\neg Q \vee R)$.

Theorem. Each formula has a disjunctive normal form and a conjunctive normal form.

Here is how we can build a disjuncvtive normal form. Consider the truth table for a given formula. For each truth value T in the table, write the conjunction that is true for the corresponding combination of truth values of the variables and is false otherwise. Then take the disjunction of the above. (The number of "pieces" joined by disjunction will be the number of T's in the truth table for the given formula.)

For example, $P \to Q$ is true for three combinations of the truth values of (P, Q): (T, T), (F, T), and (F, F). So it can be written in disjunctive normal form as a disjunction of three "pieces": $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$.

The biconditional $P \leftrightarrow Q$ is true two combinations of the truth values (P, Q): (T, T) and (F, F). So it can be written in disjunctive normal form as a disjunction of two "pieces": $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.

Similarly, $P \oplus Q$ is also true two combinations of the truth values (P, Q): (T, F) and (F, T). So it can be written in disjunctive normal form as a disjunction of two "pieces": $(P \wedge \neg Q) \vee (\neg P \wedge Q)$.

Nand opeartion.

Nand (aka alternative denial, and aka Sheffer stroke) is an operation denoted by $\uparrow$ and defined by the following truth table:

P	Q	$P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

Notice that $P \uparrow Q \equiv \neg(P \wedge Q)$, so, $P \uparrow Q$ means "not both P and Q."

Since $P \uparrow Q$ is true for three combinations of the truth values of (P, Q), namely, for (T, F), (F, T), and (F, F), it can be written in disjunctive normal form as $(P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$.

Observations and questions.

Observations:

1. Any operation (or formula) can be written in terms of $\wedge$, $\vee$, and $\neg$.
2. Since $\wedge$ can be defined in terms of $\vee$ and $\neg$, any operation can be defined in terms of these two.
   For example, since $A \wedge B \equiv \neg(\neg A \vee \neg B)$,

$$\begin{aligned} P \leftrightarrow Q &\equiv (P \wedge Q) \vee (\neg P \wedge \neg Q) \\ &\equiv (\neg(\neg P \vee \neg Q)) \vee (\neg(\neg\neg P \vee \neg\neg Q)) \\ &\equiv (\neg(\neg P \vee \neg Q)) \vee (\neg(P \vee Q)). \end{aligned}$$

3. Since $\vee$ can be defined in terms of $\wedge$ and $\neg$, any operation can be defined in terms of these two as well.
   For example, since $A \vee B \equiv \neg(\neg A \wedge \neg B)$,

$$\begin{aligned} P \leftrightarrow Q &\equiv (P \wedge Q) \vee (\neg P \wedge \neg Q) \\ &\equiv \neg(\neg(P \wedge Q) \wedge \neg(\neg P \wedge \neg Q)). \end{aligned}$$

Questions:

1. Can $\neg$ be defined in terms of $\wedge$ and $\vee$?
2. Can $\wedge$ and $\vee$ be defined in terms of $\rightarrow$ and $\neg$?
3. Can any formula be expressed in terms of just $\wedge$? Just $\vee$? Just $\neg$? Just $\rightarrow$? Just $\leftrightarrow$? Just $\oplus$?
4. Can any formula be expressed in terms of just one operation? If so, in terms of which one(s)?