Properties of quantified statements

We say that two statements with quantifiers (and propositional functions in one ore more variables) are logically equivalent if for any choice of the domain for each variable and any choice of the propositional functions, they have the same truth value. For example, we have seen before that $\neg\forall x P(x)$ and $\exists x \neg P(x)$ are logically equivalent; also, $\neg\exists x P(x)$ and $\exists x \forall P(x)$ are logically equivalent.

Note that if we take any logical equivalence in classical logic, replace the propositional variables with propositional functions, and put any quantifiers in front, we will get logically equivalent statements as well. For example, since $\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$ (this is De Morgan's Law), we have that $\forall x \exists y \neg(P(x,y) \vee Q(x,y))$ and $\forall x \exists y(\neg P(x,y) \wedge \neg Q(x,y))$ are logically equivalent as well.

Below we will consider some less obvious pairs of quantified statements.

For each of the following pairs of quantified statements determine whether they are logically equivalent. (For those that are not logically equivalent, we gave examples of the domain(s), P, and/or Q, such that the two statements have different truth values.)

1. (a) $\forall x(\neg P(x))$ and $\neg(\forall x P(x))$
   No.

   (b) $\forall x(P(x) \vee Q(x))$ and $(\forall x P(x)) \vee (\forall x Q(x))$
   No.

   (c) $\forall x(P(x) \wedge Q(x))$ and $(\forall x P(x)) \wedge (\forall x Q(x))$
   Yes.

   (d) $\forall x(P(x) \to Q(x))$ and $(\forall x P(x)) \to (\forall x Q(x))$
   No.

   (e) $\forall x(P(x) \leftrightarrow Q(x))$ and $(\forall x P(x)) \leftrightarrow (\forall x Q(x))$
   No.

   (f) $\exists x(\neg P(x))$ and $\neg(\exists x P(x))$
   No.

   (g) $\exists x(P(x) \vee Q(x))$ and $(\exists x P(x)) \vee (\exists x Q(x))$
   Yes.

   (h) $\exists x(P(x) \wedge Q(x))$ and $(\exists x P(x)) \wedge (\exists x Q(x))$
   No.

(i) $\exists x(P(x) \to Q(x))$ and $(\exists x P(x)) \to (\exists x Q(x))$
No.

(j) $\exists x(P(x) \leftrightarrow Q(x))$ and $(\exists x P(x)) \leftrightarrow (\exists x Q(x))$
No.

2. (a) $\forall x(P(x) \lor \forall y Q(x,y))$ and $\forall x \forall y(P(x) \lor Q(x,y))$
Yes.

(b) $\forall x(P(x) \land \forall y Q(x,y))$ and $\forall x \forall y(P(x) \land Q(x,y))$
Yes.

(c) $\exists x(P(x) \lor \exists y Q(x,y))$ and $\exists x \exists y(P(x) \lor Q(x,y))$
Yes.

(d) $\exists x(P(x) \land \exists y Q(x,y))$ and $\exists x \exists y(P(x) \land Q(x,y))$
Yes.

(e) $\forall x(P(x) \leftrightarrow \exists y Q(x,y))$ and $\forall x \exists y(P(x) \leftrightarrow Q(x,y))$
No.