MATH 110 **Lecture notes – 3**

Expressing some operations in terms of others revisited.

Recall the following from a previous lecture.

From the six operations $\neg$, $\wedge$, $\vee$, $\oplus$, $\rightarrow$, $\leftrightarrow$, some operations can be expressed in terms of others. For example,

$P \rightarrow Q \equiv \neg P \vee Q$.

Also, it can be checked using the truth tables that

$P \wedge Q \equiv \neg(\neg P \vee \neg Q)$,

$P \vee Q \equiv \neg(\neg P \wedge \neg Q)$,

$P \oplus Q \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$,

$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$.

Observations made earlier:

1. Any operation can be defined in terms of $\wedge$, $\vee$, and $\neg$.
2. Since $\wedge$ can be defined in terms of $\vee$ and $\neg$, any operation can be defined in terms of these two.
3. Since $\vee$ can be defined in terms of $\wedge$ and $\neg$, any operation can be defined in terms of these two as well.

Old questions and new answers:

1. Can $\neg$ be defined in terms of $\wedge$ and $\vee$?

   Answer: no. If this were possible, we would have an expression that contains only variables, $\wedge$, and $\vee$, and is logically equvalent to $\neg P$. However, when constructing a truth table for such an expression, we would only have the value T in the first line, where each variable has the value T. So, it is not possible to get an F in that line, therefore the expression cannot be logically equivalent to $\neg P$.
2. Can $\wedge$ and $\vee$ be defined in terms of $\rightarrow$ and $\neg$? If so, how? If not, explain why not.

   Answer: yes. Since $P \rightarrow Q \equiv \neg P \vee Q$, replacing P with $\neg P$ and eliminating the double negation, we have:

$$P \vee Q \equiv \neg P \to Q.$$

Applying negation to both sides of this gives

$$\neg(P \vee Q) \equiv \neg(\neg P \to Q).$$

Using DeMorgan's law,

$$\neg P \wedge \neg Q \equiv \neg(\neg P \to Q).$$

Finally, replace P with $\neg P$ and Q with $\neg Q$, and eliminate the double negation to obtain:

$$P \wedge Q \equiv \neg(P \to \neg Q).$$

3. Can any formula be expressed in terms of just $\wedge$? Just $\vee$? Just $\neg$? Just $\to$? Just $\leftrightarrow$? Just $\oplus$?

   Answer: no.

   - $\neg$ is insufficient because it cannot connect two variables.
   - $\wedge$, $\vee$, $\to$, and $\leftrightarrow$ always will give the truth value T when each variable has the value T, therefore cannot express negation.
   - $\oplus$ will always give the value F when each variable has the value F, therefore cannot express $\leftrightarrow$.

4. Can any formula be expressed in terms of just one operation? If so, in terms of which one(s)?
   Answer: yes. There are two such operations, namely,

$$P \uparrow Q = \neg(P \wedge P)$$

   and

$$P \downarrow Q = \neg(P \vee Q).$$

   First let's show that these operations $\uparrow$ and $\downarrow$ are the only binary operations that could possibly be capable of expressing all other operations.

   - To express negation, the value of the operation for P =T and Q =T must be F.

P	Q	P operation Q
T	T	F
T	F	
F	T	
F	F	

- To express biconditional, the value of the operation for $P=$F and $Q=$F must be T.

P	Q	P operation Q
T	T	F
T	F	
F	T	
F	F	T

- If the values of the operation at $P=$T, $Q=$F and at $P=$F, $Q=$T are T and F respectively, then the operation is equivalent to $\neg Q$, while if the values of the operation at $P=$T, $Q=$F and at $P=$F, $Q=$T are F and T respectively, then the operation is equivalent to $\neg P$. We already know that $\neg$ cannot express other operations.
- Thus these two values should be either both T or both F. In the first case we get $P \uparrow Q$ (called nand, or alternative denial), and in the second we get $P \downarrow Q$ (called nor, or joint denial):

P	Q	$P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

P	Q	$P \downarrow Q$
T	T	F
T	F	F
F	T	F
F	F	T

Next we will show that all other operations can be expressed in terms of $\uparrow$.
Observe that $P \uparrow P \equiv \neg(P \wedge P) \equiv \neg P$, so

$$\neg P \equiv P \uparrow P.$$

Then,

$$\begin{aligned} P \wedge Q &\equiv \neg(P \uparrow Q) \\ &\equiv (P \uparrow Q) \uparrow (P \uparrow Q) \end{aligned}$$

and

$$\begin{aligned} P \vee Q &\equiv \neg((\neg P) \wedge (\neg Q)) \\ &\equiv \neg((P \uparrow P) \wedge (Q \uparrow Q)) \\ &\equiv \neg\big(((P \uparrow P) \uparrow (Q \uparrow Q)) \uparrow ((P \uparrow P) \uparrow (Q \uparrow Q))\big) \\ &\equiv \big(((P \uparrow P) \uparrow (Q \uparrow Q)) \uparrow ((P \uparrow P) \uparrow (Q \uparrow Q))\big) \uparrow \\ &\big(((P \uparrow P) \uparrow (Q \uparrow Q)) \uparrow ((P \uparrow P) \uparrow (Q \uparrow Q))\big). \end{aligned}$$

Notice that

$$(A \uparrow A) \uparrow (A \uparrow A) \equiv \neg\neg A \equiv A,$$

so the above can be simplified:

$$P \vee Q \equiv (P \uparrow P) \uparrow (Q \uparrow Q).$$

Equivalently, using $P \wedge Q \equiv \neg(P \uparrow Q)$, we could do the following:

$$\begin{aligned} P \vee Q &\equiv \neg((\neg P) \wedge (\neg Q)) \\ &\equiv \neg(\neg((\neg P) \uparrow (\neg Q))) \\ &\equiv (\neg P) \uparrow (\neg Q) \\ &\equiv (P \uparrow P) \uparrow (Q \uparrow Q). \end{aligned}$$

Also,

$$\begin{aligned} P \to Q &\equiv \neg P \vee Q \\ &\equiv \neg(P \wedge \neg Q) \\ &\equiv \neg(P \wedge (Q \uparrow Q)) \\ &\equiv \neg((P \uparrow (Q \uparrow Q)) \uparrow (P \uparrow (Q \uparrow Q))) \\ &\equiv ((P \uparrow (Q \uparrow Q)) \uparrow (P \uparrow (Q \uparrow Q))) \uparrow ((P \uparrow (Q \uparrow Q)) \uparrow (P \uparrow (Q \uparrow Q))) \\ &\equiv P \uparrow (Q \uparrow Q). \end{aligned}$$

Equivalently, using $P \uparrow A = \neg(P \wedge A)$, we could just do

$$\begin{aligned} P \to Q &\equiv \neg P \vee Q \\ &\equiv \neg(P \wedge \neg Q) \\ &\equiv \neg(P \wedge (Q \uparrow Q)) \\ &\equiv P \uparrow (Q \uparrow Q). \end{aligned}$$

Remark. It can be shown that $P \to Q \equiv P \uparrow (P \uparrow Q)$ also, so $P \uparrow (Q \uparrow Q) \equiv P \uparrow (P \uparrow Q)$ is an identity for nand.
Next,

$$\begin{aligned} P \leftrightarrow Q &\equiv (P \to Q) \wedge (Q \to P) \\ &\equiv (P \uparrow (Q \uparrow Q)) \wedge (Q \uparrow (P \uparrow P)) \\ &\equiv ((P \uparrow (Q \uparrow Q)) \uparrow (Q \uparrow (P \uparrow P))) \uparrow ((P \uparrow (Q \uparrow Q)) \uparrow (Q \uparrow (P \uparrow P))). \end{aligned}$$

Remark. It can also be shown that $P \leftrightarrow Q \equiv (P \uparrow Q) \uparrow ((P \uparrow P) \uparrow (Q \uparrow Q))$.

Finally,

$$\begin{aligned} P \oplus Q &\equiv \neg(P \leftrightarrow Q) \\ &\equiv \neg((P \to Q) \wedge (Q \to P)) \\ &\equiv (P \to Q) \uparrow (Q \to P) \\ &\equiv (P \uparrow (Q \uparrow Q)) \uparrow (Q \uparrow (P \uparrow P)). \end{aligned}$$

Exercise: express $\neg$, $\wedge$, $\vee$, $\to$, $\leftrightarrow$, and $\oplus$ in terms of $\downarrow$.

Some properties of $\uparrow$ (where T denotes True and F denotes False):

- $P \uparrow Q \equiv Q \uparrow P$
- $P \uparrow T \equiv P \uparrow P$
- $P \uparrow F \equiv T$
- $P \uparrow (P \uparrow P) \equiv T$
- $P \uparrow (P \uparrow Q) \equiv P \uparrow (Q \uparrow Q)$
- $(P \uparrow P) \uparrow (P \uparrow P) \equiv P$

Also, observe that

$P \downarrow Q \equiv ((P \uparrow P) \uparrow (Q \uparrow Q)) \uparrow ((P \uparrow P) \uparrow (Q \uparrow Q))$