Interpretations of sormulas in sets

Interpretation.

Definition.
Suppose we have a set U (we think of it as the universal set) and a set of propositional variables such as $\{P, Q, R\}$. An interpretation of formulas (or expresssions, or compound statements) in U is a function f from the set of formulas in these variables to the power set of U (the set of all subsets of U) that satisfies the following properties: for any formulas F_1 and F_2,

1. $f(F_1 \vee F_2) = f(F_1) \cup f(F_2)$,
2. $f(F_1 \wedge F_2) = f(F_1) \cap f(F_2)$,
3. $f(\neg F_1) = \overline{f(F_1)}$.

Note that this function is completely determined by its values on the propositional variables.

Example.
Let $U = \{1, 2, 3, 4, 5\}$ and the set of variables be $\{P, Q, R\}$. Suppose
$f(P) = \{1, 2, 3\} = A$, $f(Q) = \{1, 2\} = B$, and $f(R) = \{1, 3, 4\} = C$. Then:
$f(P \vee Q) = A \cup B = \{1, 2, 3\}$,
$f(\neg R) = \overline{C} = \{2, 5\}$,
$f((P \vee Q) \wedge \neg R) = f(P \vee Q) \cap f(\neg R) = \{2\}$,
and so on. Each formula in P, Q, R, gets assigned a subset of U. This assignment of one subset of U to each formula is an interpretation of formulas in the set U.

Correspondence between lines in the truth table and regions in the Venn diagram.

Note: since F is used to denote the value False, to avoid confusion, we will always have indices for our formulas, e.g. F_1, F_2, etc. (Also note that different fonts are used for F (False) and F_1 (a formula).

Given values (i.e. sets) of the propositional variables, e.g. $f(P) = A$, $f(Q) = B$, etc., and a formula F_1 in these propositional variables, constructing the Venn diagram for $f(F_1)$ mimics constructing a truth table for F_1. More precisely, we can write the formula F_1 in the standard form using disjunction, conjunction, and negation, namely, write F_1 as the disjunction of expressions representing lines in the truth table where F_1 has the truth value T. Notice that each line corresponds to a region in the Venn diagram, and $f(F_1)$ is the union of those regions corresponding to the lines where F_1 is T.

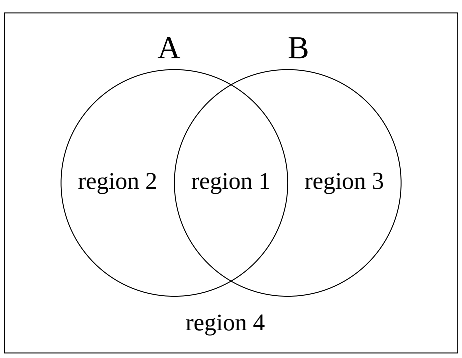
Example.
Let $f(P) = A$ and $f(Q) = B$. We will draw a Venn diagram for $P \to Q$. First write $P \to Q$ as described above:

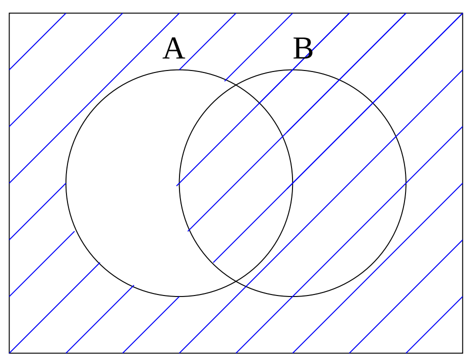
$$P \to Q \equiv (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q).$$

Here, the compound statements $P \wedge Q$, $\neg P \wedge Q$, and $\neg P \wedge \neg Q$ describe the three lines in the truth table where $P \to Q$ has the value of T:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Then,
$f(P \wedge Q) = A \cap B$ (region 1 in the Venn diagram below),
$f(\neg P \wedge Q) = \overline{A} \cap B$ (region 3 in the Venn diagram below),
$f(\neg P \wedge \neg Q) = \overline{A} \cap \overline{B}$ (region 4 in the Venn diagram below).

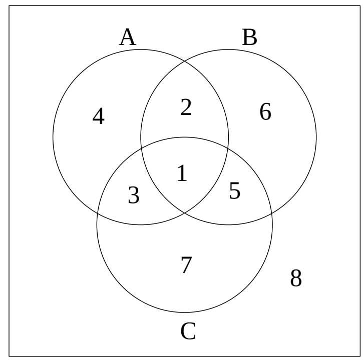




Therefore $f((P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q))$ is the union of regions 1, 3, and 4.

Similarly, if we have 3 variables, the truth table has 8 lines and the Venn diagram has 8 regions, with each region corresponding to one line:

P	Q	R	corresponding region
T	T	T	1
T	T	F	2
T	F	T	3
T	F	F	4
F	T	T	5
F	T	F	6
F	F	T	7
F	F	F	8



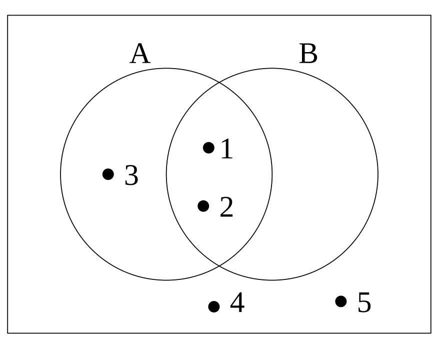
In general, for n variables, there are 2^n lines in the truth table and 2^n regions in the Venn diagram that correspond to those lines.

Observation.

1. If a formula (compound statement) F_1 is a tautology, then $f(F_1) = U$ for any interpretation (since all the regions in the Venn diagram will be shaded).
2. If a formula (compound statement) F_1 is a contradiction, then $f(F_1) = \emptyset$ for any interpretation (since none of the regions in the Venn diagram will be shaded).

Example.

Let $U = \{1, 2, 3, 4, 5\}$ and the set of variables be $\{P, Q, R\}$. Suppose $f(P) = \{1, 2, 3\} = A$, $f(Q) = \{1, 2\} = B$.



The formulas $P \vee \neg P$ and $P \wedge \neg P$ are a tautology and a contradiction, respectively, therefore their image under f must be U and $\emptyset$, respectively.

Indeed,

$$\begin{aligned} f(P \vee \neg P) &= f(P) \cup f(\neg P) \\ &= f(P) \cup \overline{f(P)} \\ &= \{1, 2, 3\} \cup \{4, 5\} \\ &= \{1, 2, 3, 4, 5\} \\ &= U \end{aligned}$$

and

$$\begin{aligned} f(P \wedge \neg P) &= f(P) \cap f(\neg P) \\ &= f(P) \cap \overline{f(P)} \\ &= \{1, 2, 3\} \cup \{4, 5\} \\ &= \emptyset \end{aligned}$$

However, **warning:** sometimes $f(F_1) = U$, but F_1 is not a tautology, or $f(F_1) = \emptyset$, but F_1 is not a contradiction. For example, for the above interpretation,

$$\begin{aligned} f((P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)) &= f(P \wedge Q) \cup f(P \wedge \neg Q) \cup f(\neg P \wedge \neg Q) \\ &= \{1, 2\} \cup \{3\} \cup \{4, 5\} \\ &= \{1, 2, 3, 4, 5\} \\ &= U \end{aligned}$$

and

$$f(\neg P \wedge Q) = \emptyset,$$

even though $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is not a tautology and $\neg P \wedge Q$ is not a contradiction.

The reason for this happening is that region 3, corresponding to the scenario $\neg P \wedge Q$ (line 3 of the truth table, where $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is false and $\neg P \wedge Q$ is true), is empty for our interpretation.