Interpretations of formulas in sets continued. Formulas valid in sets.

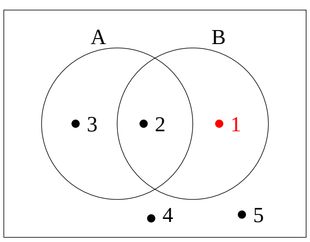
Theorem. Let U be any nonempty set. If a formula F_1 is not a tautology, then there exists an interpretation f in the set U such that $f(F_1) \neq U$.

Idea of proof. If F_1 is not a tautology, then it has the value of F (false) for at least one combination of truth values of its components (propositional variables). Since U is nonempty, it contains at least one element, say, x. Choose an interpretation such that x is in the region corresponding to the combination for which F_1 is false. That is, if a certain variable, say, P, has the value T (true) in this combination, then choose $f(P)$ to contain x. Otherwise, choose $f(P)$ not to contain x. Then $f(F_1)$ does not contain x, so $f(F_1) \neq U$.

Example. Consider $U = \{1, 2, 3, 4, 5\}$ and $F_1 = (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$. Then F_1 is not a tautology. Namely, F_1 has the value F (false) when P is false and Q is true.

P	Q	F_1
T	T	T
T	F	T
F	T	F
F	F	T

Pick any element of U, say, 1, and place it in the region corresponding to the combination where F_1 is false. The other elements can be placed in any regions. For example,



So, we define $f(P) = A = \{2, 3\}$, $f(Q) = B = \{1, 2\}$, then $f(P \wedge Q) = \{2\}$, $f(P \wedge \neg Q) = \{3\}$, $f(\neg P \wedge \neg Q) = \{4, 5\}$. So $f(F_1) = \{2, 3, 4, 5\} \neq U$.

Def. Let U be any set, and let F_1 be a formula. We say that F_1 is valid in U if for any interpretation f of formulas in U we have $f(F_1) = U$.

Theorem. Let F_1 be a formula. Then the following statements are equivalent, i.e. they are either all true or all false.

1. F_1 is a tautology,
2. F_1 is valid in any set U,
3. F_1 is valid in some nonempty set U.