Math 111 May 15, 2006
Final Exam – Solutions

1. Prove that for every odd integer n, $6n^2 + 5n + 4$ is odd.

   If n is odd, then $n = 2k+1$ for some $k \in \mathbb{Z}$. Then $6n^2+5n+4 = 6(2k+1)^2+5(2k+1)+4 = 6(2k+1)^2 + 10k + 5 + 4 = 6(2k+1)^2 + 10k + 9 = 2(3(2k+1)^2 + 5k + 4) + 1$. *Since* $3(2k+1)^2 + 5k + 4 \in \mathbb{Z}$, $6n^2 + 5n + 4$ *is odd.*

2. Make truth tables for the following compound statements.

   The truth tables are shown below.

   (a) $Q \vee (R \wedge S)$

Q	R	S	$R \wedge S$	$Q \vee (R \wedge S)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

   (b) $(P \wedge (P \iff Q)) \wedge \sim Q$

P	Q	$P \iff Q$	$P \wedge (P \iff Q)$	$\sim Q$	$(P \wedge (P \iff Q)) \wedge \sim Q$
T	T	T	T	F	F
T	F	F	F	T	F
F	T	F	F	F	F
F	F	T	F	T	F

3. Provide counterexamples to the following proposed (but false) statements.

   (a) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (x > 1 \wedge y > 0) \implies y^x > x$.
   Let $x = 2$ and $y = 1$. Then $y^x = 1$, so $y^x \not> x$.

   (b) For all positive integers x, $x^2 - x + 11$ is a prime number.
   Let $x = 11$, then $x^2 - x + 11 = 11^2 - 11 + 11 = 11^2 = 11 \cdot 11$ is not prime.

4. A sequence $\{x_n\}$ is defined recursively by $x_1 = 1$, $x_2 = 2$, and $x_n = x_{n-1} + 2x_{n-2}$ for $n \geq 3$. Conjecture a formula for x_n and verify that your conjecture is correct.

   First we find the first few terms: $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, $x_4 = 8$, $x_5 = 16$. It appears that $x_n = 2^{n-1}$.

   We will prove this conjecture by Strong Mathematical Induction.

   Basis step: if $n = 1$, then $x_1 = 2^0$ is true.

   Inductive step: assume that $x_i = 2^{i-1}$ for all i such that $1 \leq i \leq k$ for some $k \in \mathbb{N}$. We will prove that $x_{k+1} = 2^k$.

   If $k = 1$, then $x_{k+1} = x_2 = 2 = 2^1$ is true.

   If $k \geq 2$, then $x_{k+1} = x_k + 2x_{k-1} = 2^{k-1} + 2 \cdot 2^{k-2} = 2^{k-1} + 2^{k-1} = 2^k$.

5. A relation R is defined on $\mathbb{Z}$ by $x\ R\ y$ if $x \cdot y \geq 0$. Prove or disprove the following:

   (a) R is reflexive,
   For any $x \in \mathbb{Z}$, $x \cdot x \geq 0$, so $x\ R\ x$. Thus R is reflexive.

   (b) R is symmetric,
   If $x\ R\ y$, then $x \cdot y \geq 0$. Then $y \cdot x \geq 0$, so $y\ R\ x$. Thus R is symmetric.

   (c) R is transitive.
   Since $-1 \cdot 0 \geq 0$ and $0 \cdot 1 \geq 0$, but $-1 \cdot 1 \not\geq 0$, we have that $-1\ R\ 0$, $0\ R\ 1$, but $-1\ \not R\ 1$. Thus R is not transitive.

6. Let A, B, and C be sets.

   (a) Prove that $A \subseteq B$ iff $A - B = \emptyset$.
   ($\Rightarrow$) We will prove this direction by contrapositive. Let $A - B \neq \emptyset$. Then there exists $x \in A - B$. Since $x \in A$ and $x \notin B$, it follows that $A \not\subseteq B$.
   ($\Leftarrow$) We will prove this by contrapositive again. Let $A \not\subseteq B$. Then there exists $x \in A$ such that $x \notin B$. Then $x \in A - B$, therefore $A - B \neq \emptyset$.

   (b) Prove that if $A \subseteq B \cup C$ and $A \cap B = \emptyset$, then $A \subseteq C$.
   Let $x \in A$. Since $A \subseteq B \cup C$, $x \in B \cup C$. Therefore $x \in B$ or $x \in C$. Since $A \cap B = \emptyset$ and $x \in A$, $x \notin B$. Thus $x \in C$.

7. Prove that $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x + 4 & \text{if } x \leq -2 \\ -x & \text{if } -2 < x < 2 \\ x - 4 & \text{if } x \geq 2 \end{cases}$$

   is onto $\mathbb{R}$ but not one-to-one. (*Hint:* Try to graph this function; this will help you see how to prove what you need to prove.)

   The function f is not one-to-one because $f(0) = 0 = 4 - 4 = f(4)$ but $0 \neq 4$.

   It remains to show that f is onto. Let $y \in \mathbb{R}$. We will consider the following two cases.

   Case I: $y \geq 0$. Let $x = y + 4$, then $x \geq 2$, so $f(x) = x - 4 = y + 4 - 4 = y$.

   Case II: $y < 0$. Let $x = y - 4$, then $x \leq -2$, so $f(x) = x + 4 = y - 4 + 4 = y$.

Extra Credit

Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function given by $f((m, n)) = 2^{m-1}(2n - 1)$. Is f one-to-one? Is f onto?

We will prove that f is both one-to-one and onto.

Let $f((m, n)) = f((p, q))$. Since $2n - 1$ and $2q - 1$ are odd, the highest powers of 2 that divide $f((m, n))$ and $f((p, q))$ are 2^{m-1} and 2^{p-1} respectively. Since $f((m, n)) = f((p, q))$, $2^{m-1} = 2^{p-1}$. It follows that $m - 1 = p - 1$, so $m = p$. Since 2^{m-1} and 2^{p-1} are powers of two, the largest odd numbers that divide $f((m, n))$ and $f((p, q))$ are $2n - 1$ and $2p - 1$ respectively, so we also have $2n - 1 = 2p - 1$. It follows that $n = p$. So $(m, n) = (p, q)$. Thus f is one-to-one.

Let $r \in \mathbb{N}$. Let 2^k be the largest power of 2 that divides r. Then $r = 2^k l$ where $l \in \mathbb{N}$, l is odd. Then $l = 2x + 1$ for some $x \in \mathbb{Z}$. Let $m = k + 1$ and $n = x + 1$. Then $k = m - 1$ and $l = 2x + 1 = 2(n - 1) + 1 = 2n - 1$. Therefore $r = 2^{m-1}(2n - 1)$. Since $k \in \mathbb{Z}$ and $k \geq 0$, we have $m \in \mathbb{N}$ and since $l > 1$, $x > 0$, so $n \in \mathbb{N}$. Thus the number r is in the image, so f is onto.