Homework 4 - Solutions

1. For any $x \in \mathbb{R}$, $x^2 + 2x + 4 = x^2 + 2x + 1 + 3 = (x+1)^2 + 3 \geq 0 + 3 = 3$. Since the conclusion is true for all x, the implication is true. (This is a trivial proof.)

2. For any $n \in \mathbb{Z}$, $n^2 - 2n + 5 = n^2 - 2n + 1 + 4 = (n-1)^2 + 4 \geq 4 > 3$, therefore $n^2 - 2n + 5 \not\leq 3$. Since the hypothesis is false for all n, the implication is true. (This is a vacuous proof.)

3. We have $2n^2 - 8n + 10 = 2(n^2 - 4n + 5)$. Since n is an integer, $n^2 - 4n + 5$ is an integer, and thus $2n^2 - 8n + 10$ is even.

3.2. If x is an even integer, then $x = 2y$ for some integer y. Then $5x - 3 = 5 \cdot 2y - 3 = 10y - 3 = 10y - 4 + 1 = 2(5y - 2) + 1$. Since $5y - 2$ is an integer, $5x - 3$ is an odd integer.

3.4. Proof by contrapositive. Suppose x is not even, i.e. is odd. Then $x = 2y + 1$ for some integer y, and $7x + 5 = 7(2y + 1) + 5 = 14y + 12 = 2(7y + 6)$. Since $7y + 6$ is an integer, $7x + 5$ is an even integer, therefore is not odd. This proves that if $7x + 5$ is odd, then x is even.

3.6. First we will prove that if $5x - 11$ is even, then x is odd. We will prove this by contrapositive. Assume x is not odd, i.e. is even. Then $x = 2y$ for some integer y, and $5x - 11 = 5(2y) - 11 = 10y - 11 = 10y - 12 + 1 = 2(5y - 6) + 1$. Since $5y - 6$ is an integer, $5x - 11$ is odd, therefore is not even.

Next we prove that if x is odd, then $5x - 11$ is even. This part we will prove directly. If x is odd, then $x = 2y + 1$ for some integer y, and $5x - 11 = 5(2y + 1) - 11 = 10y - 6 = 2(5y - 3)$. Since $5y - 3$ is an integer, $5x - 11$ is even.

3.8. Lemma. Let $x \in \mathbb{Z}$. If $7x + 4$ is even, then x is even.

Proof of lemma (by contrapositive). Suppose x is not even, i.e. is odd. Then $x = 2y + 1$ for some integer y, and $7x + 4 = 7(2y + 1) + 4 = 14y + 11 = 14y + 10 + 1 = 2(7y + 5) + 1$. Since $7y + 5$ is an integer, $7x + 4$ is odd, i.e. is not even.

Proof of the result. If $7x + 4$ is even, then by the above lemma x is even. Therefore $x = 2y$ for some integer y, and $3x - 11 = 3(2y) - 11 = 6y - 11 = 6y - 12 + 1 = 2(3y - 6) + 1$. Since $3y - 6$ is an integer, $3x - 11$ is odd.