MATH 111 **Fall 2007**

Homework 5 - Solutions

3.12. Proof by cases.

Case I: the number n is even. Then $n = 2m$ for some $m \in \mathbb{Z}$, and $n^2 - 3n + 9 = (2m)^2 - 3(2m) + 9 = 4m^2 - 6m + 9 = 2(2m^2 - 3m + 4) + 1$. Since $2m^2 - 3m + 4 \in \mathbb{Z}$, $n^2 - 3n + 9$ is odd.

Case II: the number n is odd. Then $n = 2m + 1$ for some $m \in \mathbb{Z}$, and $n^2 - 3n + 9 = (2m + 1)^2 - 3(2m + 1) + 9 = 4m^2 + 4m + 1 - 6m - 3 + 9 = 4m^2 - 2m + 7 = 2(2m^2 - m + 3) + 1$. Since $2m^2 - m + 3 \in \mathbb{Z}$, $n^2 - 3n + 9$ is odd.

3.14. Proof by contrapositive. Suppose that it is not the case that both x and y are odd, i.e. at least one of them is even. Without loss of generality we can assume that x is even. Then $x = 2k$ for some $k \in \mathbb{Z}$, and $xy = (2k)y = 2(ky)$. Since $ky \in \mathbb{Z}$, xy is even, i.e. not odd.

3.16. We will prove the statement by contrapositive, i.e. we will prove that if x and y are not of the same parity, then $3x + 5y$ is odd.

Case I: x is even and y is odd. Then $x = 2m$ and $y = 2n + 1$ for some $m, n \in \mathbb{Z}$. Then $3x + 5y = 3(2m) + 5(2n + 1) = 6m + 10n + 5 = 6m + 10n + 4 + 1 = 2(3m + 5n + 2) + 1$. Since $3m + 5n + 2 \in \mathbb{Z}$, $3x + 5y$ is odd.

Case II: x is odd and y is even. Then $x = 2m + 1$ and $y = 2n$ for some $m, n \in \mathbb{Z}$. Then $3x + 5y = 3(2m + 1) + 5(2n) = 6m + 3 + 10n = 6m + 10n + 2 + 1 = 2(3m + 5n + 1) + 1$. Since $3m + 5n + 1 \in \mathbb{Z}$, $3x + 5y$ is odd.

3.20. Proof by cases.

Case I: the number x is even. Then $x = 2y$ for some $y \in \mathbb{Z}$. Therefore $3x + 1 = 3(2y) + 1 = 2(3y) + 1$ and $5x + 2 = 5(2y) + 2 = 10y + 2 = 2(5y + 1)$. Since $3y$ and $5y + 1$ are integers, $3x + 1$ is odd and $5x + 2$ is even, so they are of opposite parity.

Case II: the number x is odd. Then $x = 2y + 1$ for some $y \in \mathbb{Z}$. Therefore $3x + 1 = 3(2y + 1) + 1 = 6y + 4 = 2(3y + 2)$ and $5x + 2 = 5(2y + 1) + 2 = 10y + 7 = 2(5y + 3) + 1$. Since $3y + 2$ and $5y + 3$ are integers, $3x + 1$ is even and $5x + 2$ is odd, so they are of opposite parity.

3.22. The converse of the result is proved. The result stated is not proved because the converse of an implication is not logically equivalent to the implication itself.

4.2. If $a|b$ and $b|a$, then by definition $b = ac$ for some $c \in \mathbb{Z}$ and $a = bd$ for some $d \in \mathbb{Z}$. Then $a = bd = acd$. Since $a \neq 0$, it follows that $cd = 1$. The only pairs of integers whose product is 1 are $1 \cdot 1 = 1$ and $(-1) \cdot (-1) = 1$. If $c = d = 1$, then $a = b$. If $c = d = -1$, then $a = -b$.

4.4. If $3 \nmid x$, then either $x = 3k+1$ or $x = 3k+2$ for some $k \in \mathbb{Z}$. If $3 \nmid y$, then either $y = 3l+1$ or $y = 3l+2$ for some $l \in \mathbb{Z}$. Thus we have four cases.

Case I: $x = 3k+1$ and $y = 3l+1$. Then $x^2 - y^2 = (3k+1)^2 - (3l+1)^2 = 9k^2 + 6k + 1 - 9l^2 - 6l - 1 = 9k^2 + 6k - 9l^2 - 6l = 3(3k^2 + 2k - 3l^2 - 2l)$. Since $3k^2 + 2k - 3l^2 - 2l \in \mathbb{Z}$, $3 | x^2 - y^2$.

Case II: $x = 3k+1$ and $y = 3l+2$. Then $x^2 - y^2 = (3k+1)^2 - (3l+2)^2 = 9k^2 + 6k + 1 - 9l^2 - 12l - 4 = 9k^2 + 6k - 9l^2 - 12l - 3 = 3(3k^2 + 2k - 3l^2 - 4l - 1)$. Since $3k^2 + 2k - 3l^2 - 4l - 1 \in \mathbb{Z}$, $3 | x^2 - y^2$.

Case III: $x = 3k+2$ and $y = 3l+1$. Then $x^2 - y^2 = (3k+2)^2 - (3l+1)^2 = 9k^2 + 12k + 4 - 9l^2 - 6l - 1 = 9k^2 + 12k - 9l^2 - 6l + 3 = 3(3k^2 + 4k - 3l^2 - 2l + 1)$. Since $3k^2 + 4k - 3l^2 - 2l + 1 \in \mathbb{Z}$, $3 | x^2 - y^2$.

Case IV: $x = 3k+2$ and $y = 3l+2$. Then $x^2 - y^2 = (3k+2)^2 - (3l+2)^2 = 9k^2 + 12k + 4 - 9l^2 - 12l - 4 = 9k^2 + 12k - 9l^2 - 12l = 3(3k^2 + 4k - 3l^2 - 4l)$. Since $3k^2 + 4k - 3l^2 - 4l \in \mathbb{Z}$, $3 | x^2 - y^2$.