Homework 8 - Solutions

5.2. Assume that there is a smallest positive irrational number, say, x. Since $2 = \frac{2}{1} \in \mathbb{Q}$, by problem 5.8, $\frac{x}{2}$ is irrational. Also, $0 < \frac{1}{2} < 1$ implies $0 < \frac{x}{2} < x$, so x is not a smallest positive irrational number. Contradiction.

5.4. Assume that there exist odd integers a and b such that $4|(a^2 + b^2)$. Then $a = 2k + 1$, $b = 2l + 1$, and $a^2 + b^2 = 4m$ for sime $k, l, m \in \mathbb{Z}$. It follows that $(2k+1)^2 + (2l+1)^2 = 4m$. Equivalently, $4k^2 + 4k + 4l^2 + 4l + 2 = 4m$. Therefore $k^2 + k + l^2 + l + \frac{1}{2} = m$. Since $k^2 + k + l^2 + l + \frac{1}{2} \notin \mathbb{Z}$ and $m \in \mathbb{Z}$, we have a contradiction.

5.6. Assume that 1000 can be written as the sum of three integers, an enen number of which are even. We will consider two cases.

Case I: Zero of the three integers are even, i.e. all three are odd. Let $1000 = x+y+z$ where x, y, and z are odd integers. Then $x = 2k+1$, $y = 2l+1$, and $z = 2m+1$ for some $k, l, m \in \mathbb{Z}$. Then $1000 = 2k+1+2l+1+2m+1$. Dividing both sides of this equiation by 2 gives $500 = k+l+m+1.5$. Since $500 \in \mathbb{Z}$ and $k+l+m+1.5 \notin \mathbb{Z}$, we have a contradiction.

Case II: Two of the three integers are even, and one is odd. Let $1000 = x + y + z$ where x and y are even and z is odd. Then $x = 2k$, $y = 2l$, and $z = 2m + 1$ for some $k, l, m \in \mathbb{Z}$. Then $1000 = 2k + 2l + 2m + 1$. Dividing both sides of this equiation by 2 gives $500 = k+l+m+0.5$. Since $500 \in \mathbb{Z}$ and $k+l+m+0.5 \notin \mathbb{Z}$, we have a contradiction.

5.8. Assume that there exist an irrational number x and a nonzero rational number r such that $\frac{x}{r}$ is rational. Then $r = \frac{k}{l}$ and $\frac{x}{r} = \frac{m}{n}$ for some $k, l, m, n \in \mathbb{Z}$, $l \neq 0$, $n \neq 0$. It follows that $x = \frac{m}{n} r = \frac{mk}{nl}$. Since $mk, nl \in \mathbb{Z}$ and $nl \neq 0$, x is rational. We get a contradiction.

5.10. Lemma. Let $a \in \mathbb{Z}$. If $3|a^2$, then $3|a$.

Proof (by contrapositive). Let $3 \not| a$. We will show that $3 \not| a^2$.

Since $3 \not| a$, then either $a = 3k + 1$ or $a = 3k + 2$ for some integer k. In the first case, $a^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$, thus $3 \not| a^2$. In the second case, $a^2 = (3k+2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$, thus, again, $3 \not| a^2$.

Now we will prove that $\sqrt{3}$ is irrational. Suppose $\sqrt{3}$ is rational, then $\sqrt{3} = \frac{m}{n}$, where $m, n \in \mathbb{Z}$, $n \neq 0$, and m and n are relatively prime (i.e. $\frac{m}{n}$ is in lowest possible terms). Squaring both sides of the above equation gives $3 = \frac{m^2}{n^2}$, so $3n^2 = m^2$. Thus $3|m^2$. By the above lemma, $3|m$, so $m = 3k$ for some $k \in \mathbb{Z}$.

Then we have $3n^2 = 9k^2$, or $n^2 = 3k^2$. Now we see that $3|n^2$, and by the above lemma, $3|n$. Since both m and n are divisible by 3, they are not relatively prime (i.e. the fraction $\frac{m}{n}$ is not in lowest possible terms). Contradiction.

5.14. Assume that there exists a positive integer x such that $2x < x^2 < 3x$. Then $2 < x < 3$. Since there are no integers larger than 2 and smaller than 3, we have a contradiction.

5.16. Direct proof: if n is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then $7n - 5 = 7(2k + 1) - 5 = 14k + 2 = 2(7k + 1)$. Since $7k + 1 \in \mathbb{Z}$, $7n - 5$ is even.

Proof by contrapositive: if $7n - 5$ is odd, then $7n - 5 = 2k + 1$ for some $k \in \mathbb{Z}$. Then $n = 7n - 5 - 6n + 5 = 2k + 1 - 6n + 5 = 2k - 6n + 6 = 2(k - 3n + 3)$. Since $k - 3n + 3 \in \mathbb{Z}$, n is even.

Proof by contradiction: suppose n is odd and $7n - 5$ is also odd. Then $n = 2k + 1$ and $7n - 5 = 2l + 1$ for some $k, l \in \mathbb{Z}$. The first equation implies $7n = 14k + 7$. Subtracting $7n - 5 = 2l + 1$ from $7n = 14k + 7$ gives $7n - 7n + 5 = 14k + 7 - 2l - 1$, thus $5 = 14k - 2l + 6$, so $2.5 = 7k - l + 3 \in \mathbb{Z}$. Since 2.5 is not an integer, we get a contradiction.

5.20. In Case I we cannot assume that x and y are odd because it could be the case that x and z, or y and z are odd. "Without loss of generality" should be used. That is, this case should start with "two of the numbers x, y, z are odd. Without loss of generality we can assume that x and y are odd and z is even." Also, since the result mentions the parity of odd (and not even) numbers, it would be better to start Case II with "Zero (or none) of the numbers x, y, z are odd, i.e. all three are even."