MATH 111

Test 1 - Solutions

1. Let $A = \{x \in \mathbb{Z} \mid 0 \leq x \leq 3\}$ and $B = \{x \in \mathbb{Z} \mid x \leq 2\}$, and let $\mathbb{Z}$ be the universal set. Determine the following sets:

   (a) $A \cap B = \{x \in \mathbb{Z} \mid 0 \leq x \leq 2\} = \{0, 1, 2\}$

   (b) $\overline{B} = \{x \in \mathbb{Z} \mid x > 2\} = \{2, 3, 4, \ldots\}$

   (c) $A \cup \overline{B} = \{x \in \mathbb{Z} \mid x \geq 0\} = \{0, 1, 2, \ldots\}$

2. Let P and Q be propositions. Prove that the compound propositions $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ are logically equivalent.

   We construct the truth table:

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

   We see that for any combination of truth values of P and Q, $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ have the same truth values. Therefore these compound propositions are logically equivalent.

3. Let $N(x, y)$ denote "x knows y's name" where x and y are students at Fresno State. Write in words the following statements and predict their truth values. Explain your reasons!

   (a) $\forall x \forall y N(x, y)$
   Every Fresno State student knows the name of every Fresno State student. This statement is false, e.g. I do not know the name of the person who sits in front of me in Math 111 (or something like this; giving a specific example would be most convincing).

   (b) $\exists x \forall y N(x, y)$
   There is a student at Fresno State who knows the name of every Fresno State student. This statement is probably false, because there are over 20,000 students at Fresno State, and it is impossible to know that many names.

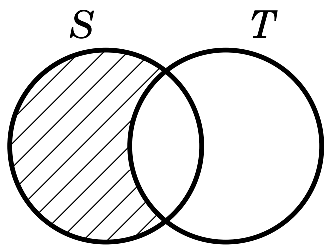
   (c) $\forall x \exists y N(x, y)$
   Every Fresno State student knows the name of at least one Fresno State student. This statement is (probably) true, since every student (hopefully) knows at least his or her own name.

   (d) $\forall x \forall y (N(x, y) \Rightarrow N(y, x))$
   If one Fresno State student knows the name of another one, then the second one knows the name of the first. This statement is false, e.g. the person who sits in front of me in Math 111 greets me by name, but I don't know his name. Or: I

know the name of the Math Club President, but I doubt she knows mine since I've just transfered to CSUF and haven't introduced myself to her yet. (or something like this; again, giving a specific example would be most convincing, although you may not be able to give one; in this case, you can just say that it is very probable that there are two such people).

Note that you were asked to *predict* the truth values. You have to explain your reasons, however, you may not be able to give rigorous proofs in some cases.

4. Let S and T be sets. Draw a Venn diagram of $S - T$.



5. Which of the following propositions can be proved using a vacuous proof? Prove it (use a vacuous proof).

   - Let $n \in \mathbb{Z}$. If $4n + 5$ is even, then $7n + 8$ is odd.
   - Let $n \in \mathbb{Z}$. If $7n + 8$ is odd, then $4n + 5$ is even.
   - Let $n \in \mathbb{Z}$. Then $4n + 5$ is even if and only if $7n + 8$ is odd.

   The first proposition can be proved using a vacuous proof, i.e. by showing that $4n+5$ is never even.
   Proof. Since $4n + 5 = 2(2n + 2) + 1$ and $2n + 2 \in \mathbb{Z}$, $4n + 5$ is odd.

6. Let $n \in \mathbb{N}$. Prove that $3n + 7$ is odd if and only if n is even.

   (⇒) We will prove this direction by contrapositive, i.e. we will prove that if n is odd, then $3n + 7$ is even. If n is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then $3n + 7 = 3(2k + 1) + 7 = 6k + 10 = 2(3k + 5)$. Since $3k + 5 \in \mathbb{Z}$, $3n + 7$ is even.

   (⇐) If n is even, then $n = 2k$ for some $k \in \mathbb{Z}$. Then $3n + 7 = 3(2k) + 7 = 6k + 7 = 2(3k + 3) + 1$. Since $3k + 3 \in \mathbb{Z}$, $3n + 7$ is odd.

7. (For extra credit) Prove that if sets A and B are disjoint, then $|\mathcal{P}(A \cup B)| = |\mathcal{P}(A)| \cdot |\mathcal{P}(B)|$. What is $|\mathcal{P}(A \cap B)|$ in this case?

   Since A and B are disjoint, $|A \cup B| = |A| + |B|$. It follows that $|\mathcal{P}(A \cup B)| = 2^{|A \cup B|} = 2^{|A|+|B|} = 2^{|A|} \cdot 2^{|B|} = |\mathcal{P}(A)| \cdot |\mathcal{P}(B)|$.

   Since $A \cap B = \emptyset$, $\mathcal{P}(A \cap B) = \{\emptyset\}$. Thus $|\mathcal{P}(A \cap B)| = 1$.