MATH 111
Test 2 (11/05/07) - Solutions

1. Let $a \in \mathbb{Z}$. Prove that if $4|a^2$, then $2|a$.

   Proof by contrapositive. Let $2 \not| a$. Then $a = 2k+1$ for some $k \in \mathbb{Z}$. Then $a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2+k) + 1$. Since $k^2 + k \in \mathbb{Z}$, $4 \not| a^2$.

2. Prove that $\sqrt[3]{2}$ is an irrational number.

   Proof by contradiction. Assume that $\sqrt[3]{2}$ is rational, then $\sqrt[3]{2} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ such that $a > 0$, $b > 0$, and a and b are relatively prime. Cubing both sides of the above equation gives $2 = \frac{a^3}{b^3}$, therefore $a^3 = 2b^3$. Since $b^3 \in \mathbb{Z}$, $2|a^3$. By the Lemma proved below, $2|a$. Then $a = 2k$ for some $k \in \mathbb{Z}$. Therefore $(2k)^3 = 2b^3$. Equivalently, $4k^3 = b^3$. Since $b^3 = 2(2k^3)$ and $2k^3 \in \mathbb{Z}$, $2|b^3$. By Lemma, $2|b$. We get a contradiction since we assumed that a and b were relatively prime.

   Lemma. Let $n \in \mathbb{Z}$. If $2|n^3$, then $2|n$.

   Proof by contrapositive. Let $2 \not| n$. Then $n = 2k+1$ for some $k \in \mathbb{Z}$. Therefore $n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$. Since $4k^3 + 6k^2 + 3k \in \mathbb{Z}$, $2 \not| n^3$.

3. Prove or disprove. The equation $x^3 + 5x + 2 = 0$ has a real solution.

   The statement is true. Let $f(x) = x^3 + 5x + 2$. Since $f(x)$ is continuous, $f(0) = 2 > 0$, and $f(-1) = -4 < 0$, by the Intermediate Value Theorem $f(x)$ has a root.

4. Prove or disprove. Let A and B be sets. Then $(A - B) \cap (A \cup B) = A$.

   The statement is false. Let $A = \{1, 2\}$, $B = \{2, 3\}$, then $A - B = \{1\}$, $A \cup B = \{1, 2, 3\}$, so $(A - B) \cap (A \cup B) = \{1\} \neq A$.

5. Prove or disprove. For any integer a there exist an integer b such that $b < a$ and $a \equiv b (\text{mod } 2)$.

   The statement is true. For any integer a, let $b = a - 2$. Then $b < a$ and $a \equiv b (\text{mod } 2)$ because $2|(a - a + 2)$.

6. (For extra credit) Prove or disprove. The number $\frac{\sqrt{2}-1}{\sqrt{2}+1}$ is irrational.

   The statement is true. We will prove it by contradiction. Assume that $\frac{\sqrt{2}-1}{\sqrt{2}+1}$ is rational. Then $\frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, $b \neq 0$. Then $b(\sqrt{2}-1) = a(\sqrt{2}+1)$. Equivalently, $b\sqrt{2} - b = a\sqrt{2} + a$, or $\sqrt{2}(b - a) = a + b$, so $\sqrt{2} = \frac{a+b}{b-a}$. Since $a + b, a - b \in \mathbb{Z}$ and $b - a \neq 0$ ($b \neq a$ since $\sqrt{2} - 1 \neq \sqrt{2} + 1$), $\sqrt{2}$ is rational. Contradiction.