Practice Test 2 - Solutions

1. Read the textbook!
2. (a) If n is an integer such that $5|(n-1)$, then $n \equiv 1 \pmod 5$. Then $n^3+n-2 \equiv 1^3+1-2 \equiv 0 \pmod 5$. This implies that $5|(n^3+n-2)$. (This is a direct proof.)
   Another proof: If n is an integer such that $5|(n-1)$, then $n-1 = 5k$ for some $k \in \mathbb{Z}$. Then $n = 5k+1$, therefore $n^3+n-2 = (5k+1)^3+(5k+1)-2 = 125k^3+75k^2+15k+1+5k+1-2 = 125k^3+75k^2+20k = 5(25k^3+15k^2+4k)$. Since $25k^3+15k^2+4k \in \mathbb{Z}$, $5|(n^3+n-2)$. (This is also a direct proof.)

   (b) Assume that $\log_3 2$ is rational. Then $\log_3 2 = \dfrac{m}{n}$ for some $m, n \in \mathbb{Z}$, $n > 0$. Then $3^{\frac{m}{n}} = 2$, so $3^m = 2^n$. Since $n > 0$, $3^m = 2^n > 1$, so $m > 0$. Since $3 \equiv 1 \pmod 2$, $3^m \equiv 1 \pmod 2$, so 3^m is odd. However, $2^n = 2 \cdot 2^{n-1}$ is even. We get a contradiction. Therefore $\log_3 2$ is irrational. (This is a proof by contradiction.)

   (c) We will prove this statement by contrapositive. Assume that n is odd. Then $n = 2k+1$ for some $k \in \mathbb{Z}$. Then $7n^2+4 = 7(2k+1)^2+4 = 7(4k^2+4k+1)+4 = 28k^2+28k+11 = 2(14k^2+14k+5)+1$. Since $14k^2+14k+5 \in \mathbb{Z}$, $7n^2+4$ is odd.

   (d) We will prove this statement by contrapositive. Assume that $x \geq 1$. Then $x^2 \geq x$ and $x^3 \geq x$. Adding these two inequalities gives $x^2+x^3 \geq 2x$, thus $2x \not> x^2+x^3$.

   (e) First we will prove that if $3|(mn)$, then $3|m$ or $3|n$. We will prove this by contrapositive, namely, we will prove that if $3 \nmid m$ and $3 \nmid n$, then $3 \nmid (mn)$. If $3 \nmid m$, then $m = 3k+1$ or $m = 3k+2$ for some $k \in \mathbb{Z}$. If $3 \nmid n$, then $n = 3l+1$ or $n = 3l+2$ for some $l \in \mathbb{Z}$. Thus we have four cases:
   <u>Case I:</u> $m = 3k+1$, $n = 3l+1$. Then $mn = (3k+1)(3l+1) = 9kl+3k+3l+1 = 3(3kl+k+l)+1$. Since $3kl+k+l \in \mathbb{Z}$, $3 \nmid (mn)$.
   <u>Case II:</u> $m = 3k+1$, $n = 3l+2$. Then $mn = (3k+1)(3l+2) = 9kl+6k+3l+2 = 3(3kl+2k+l)+2$. Since $3kl+2k+l \in \mathbb{Z}$, $3 \nmid (mn)$.
   <u>Case III:</u> $m = 3k+2$, $n = 3l+1$. Then $mn = (3k+2)(3l+1) = 9kl+3k+6l+2 = 3(3kl+k+2l)+2$. Since $3kl+k+2l \in \mathbb{Z}$, $3 \nmid (mn)$.
   <u>Case IV:</u> $m = 3k+2$, $n = 3l+2$. Then $mn = (3k+2)(3l+2) = 9kl+6k+6l+4 = 3(3kl+2k+2l+1)+1$. Since $3kl+2k+2l+1 \in \mathbb{Z}$, $3 \nmid (mn)$.
   Next we will prove that if $3|m$ or $3|n$, then $3|(mn)$. Here we have two cases:
   <u>Case I:</u> $3|m$. Then $m = 3k$ for some $k \in \mathbb{Z}$. Then $mn = 3kn$. Since $kn \in \mathbb{Z}$, $3|(mn)$.
   <u>Case II:</u> $3|n$. Then $n = 3l$ for some $l \in \mathbb{Z}$. Then $mn = m3l = 3ml$. Since

$ml \in \mathbb{Z}$, $3|(mn)$.
(This direction we proved directly.)

(f) Assume that there exist a nonzero rational number x and an irrational number y such that xy is rational. Then $x = \dfrac{k}{l}$ for some $k, l \in \mathbb{Z}$, $k \neq 0$ and $l \neq 0$, and $xy = \dfrac{m}{n}$ for some $m, n \in \mathbb{Z}$, $n \neq 0$. Then $y = \dfrac{xy}{x} = \dfrac{\frac{m}{n}}{\frac{k}{l}} = \dfrac{ml}{nk}$. Since $ml, nk \in \mathbb{Z}$ and $nk \neq 0$, y is rational. Contradiction. (This is a proof by contradiction.)

(g) We will prove this statement by contrapositive. Namely, we will assume that $a|b$ or $a|c$ and we will show that $a|(bc)$. If $a|b$, then $b = ak$ for some $k \in \mathbb{Z}$, and $bc = akc$. Since $kc \in \mathbb{Z}$, $a|(bc)$. If $a|c$, then $c = ak$ for some $k \in \mathbb{Z}$, and $bc = bak = abk$. Since $bk \in \mathbb{Z}$, $a|(bc)$.

(h) First we will prove that if $A \cap B = \emptyset$, then $(A \times B) \cap (B \times A) = \emptyset$. We will prove this by contrapositive. Assume that $(A \times B) \cap (B \times A) \neq \emptyset$. Then there exists $x \in (A \times B) \cap (B \times A)$, thus $x \in A \times B$ and $x \in B \times A$. Therefore $x = (y, z)$ where $y \in A$, $z \in B$, $y \in B$, and $z \in A$. Since $y \in A$ and $y \in B$, it follows that $A \cap B \neq \emptyset$.
Next we will prove that if $(A \times B) \cap (B \times A) = \emptyset$, then $A \cap B = \emptyset$. We will prove this by contrapositive as well. Assume that $A \cap B \neq \emptyset$, then there exists $x \in A \cap B$, i.e. $x \in A$ and $x \in B$. Then $(x, x) \in A \times B$ and $(x, x) \in B \times A$, so $(x, x) \in (A \times B) \cap (B \times A)$. Thus $(A \times B) \cap (B \times A) \neq \emptyset$.

3. (a) This statement is true. For example, if $a = -1$, then for every real number b, we have $b^2 \geq 0 \geq -1$, so $b^2 \geq a$.

(b) This statement is false. For any integer a, either $a \leq 4$ or $a \geq 5$. If $a \leq 4$, then $a^3 + 2a + 3 \leq 64 + 8 + 3 = 75 < 100$, so $a^3 + 2a + 3 \neq 100$. If $a \geq 5$, then $a^3 + 2a + 3 \geq 125 + 10 + 3 = 138 > 100$, so $a^3 + 2a + 3 \neq 100$.

(c) This statement is false. For example, if $a = -1$, then there is no integer b such that $b^2 = -1$.

(d) This statement is false. For example, $\sqrt{2} + (2 - \sqrt{2}) = 2$. We know that $\sqrt{2}$ is irrational (we proved such a theorem). The fact that $2 - sqrt2$ is irrational can be proved by contradiction. Namely, assume that $2 - sqrt2$ is rational, then $2 - sqrt2 = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$, $n \neq 0$. Then $\sqrt{2} = 2 - \frac{m}{n} = \frac{2n-m}{n}$. Since $2n - m \in \mathbb{Z}$ and $n \neq 0$, $sqrt2$ is rational. Contradiction. Finally, $2 = \frac{2}{1}$ is rational.

(e) This statement is true. Let a be any irrational number. Then $a = 1 + (a-1)$. Observe that 1 is rational, and $a - 1$ is irrational (the proof of this is similar to the proof given in previous problem, and is omitted here).

(f) This statement is true. For any sets A and B, let $C = A \cup B$. Then $A \cup C = A \cup A \cup B = A \cup B$ and $B \cup C = B \cup A \cup B = A \cup B$, so $A \cup C = B \cup C$.

(g) This statement is false. For example, if $A = \{1\}$, $B = \{2\}$, $C = \{1, 2\}$, $D = \{2, 3\}$, then $A \subset C$, $B \subset D$, and $A \cap B = \emptyset$, however, $C \cap D \neq \emptyset$.

(h) This statement if true. Suppose that $A \subset C$, $B \subset D$, $C \cap D = \emptyset$, but $A \cap B \neq \emptyset$. Then there is an element $x \in A \cap B$, so $x \in A$ and $x \in B$. Since $A \subset C$ and $B \subset D$, it follows that $x \in C$ and $x \in D$. Then $x \in C \cap D$, thus $C \cap D \neq \emptyset$. We get a contradiction.