MATH 111
Test 1 - Solutions

1. Let $\mathbb{R}$ be the universal set, and let $A = [0, 3)$ and $B = (-\infty, 2)$.

   (a) Determine and write in the interval notation the following sets:

      i. $A \cup B = (-\infty, 3)$

      ii. $\overline{B} = [2, +\infty)$

      iii. $\overline{A} \cap B = (-\infty, 0)$

   (b) How many elements does A have?

      It has infinitely many elements.

2. Let P and Q be propositions. Are compound propositions $P \Rightarrow Q$ and $P \vee \neg Q$ logically equivalent? If so, prove it. If not, provide an example of P and Q for which one of these compound propositions is true and the other one is false.

   We construct the truth table to check whether the given compound propositions are logically equivalent:

P	Q	$P \Rightarrow Q$	$\neg Q$	$P \vee \neg Q$
T	T	T	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

   We see that when P and Q have opposite truth values, $P \Rightarrow Q$ and $P \vee \neg Q$ have opposite truth values. Therefore these compound propositions are not logically equivalent.

   Example: let P denote "2 is even", which is a true proposition; let Q denote "3 is even", which is a false propositions. For these P and Q, the proposition $P \Rightarrow Q$ is false and the proposition $P \vee \neg Q$ is true.

3. Let $x \in \mathbb{R}$, and let $P(x, y)$ denote "$x \geq y + 2$". Determine the truth values of the following propositions. (Explain your answers!)

   (a) $\forall x \exists y P(x, y)$ *is true because for any x we can choose $y = x - 2$, then $x = x - 2 + 2 = y + 2$, so $x \geq y + 2$.*

   (b) $\exists y \forall x P(x, y)$ *is false. No matter what y is, not all values of x satisfy this inequality: e.g. $x = y$ doesn't satisfy $x \geq y + 2$ since it is not true that $y \geq y + 2$ (because $0 < 2$, so $y < y + 2$ for any y).*

   (c) $\exists! x P(x, 1)$ *is false because $P(x, 1)$ denotes "$x \geq 3$", and there are more than one value of x greater than or equal to 3, e.g. $x = 3$ and $x = 4$.*

4. Which of the following implications can be proved using a trivial proof? Prove it (use a trivial proof).

   - Let $x \in \mathbb{R}$. If $x^2 < -25$, then $x < -5$.

- Let $n \in \mathbb{Z}$. If $8 < n \leq 39$, then $6n + 4$ is even.
- Let $n \in \mathbb{Z}$. If n is odd, then $4n$ and $5n$ are of opposite parity.

Answer: the second implication can be proved using a trivial proof.

Proof. *Since* $6n + 4 = 2(3n + 2)$ *and* $3n + 2 \in \mathbb{Z}$, *the number* $6n + 4$ *is even.*

(Note: this is a trivial proof because we proved that the conclusion is true without assuming the hypothesis.)

5. Let $n \in \mathbb{N}$. Prove that $4n^2 - 6n - 3$ is an odd integer.

   Since $4n^2 - 6n - 3 = 4n^2 - 6n - 4 + 1 = 2(2n^2 - 3n - 2) + 1$ *and* $2n^2 - 3n - 2 \in \mathbb{Z}$, *the number* $4n^2 - 6n - 3$ *is odd.*

6. Let $n \in \mathbb{N}$. Prove that $5n + 3$ is odd if and only if n is even.

   ($\Rightarrow$) We will prove this direction by contrapositive, namely, we will prove that if n *is odd, then* $5n + 3$ *is even.*

   If n *is odd,* $n = 2k+1$ *for some* $k \in \mathbb{Z}$. *Then* $5n+3 = 5(2k+1)+3 = 10k+8 = 2(5k+4)$. *Since* $5k + 4 \in \mathbb{Z}$, *the number* $5n + 3$ *is even.*

   ($\Leftarrow$) If n *is even, then* $n = 2k$ *for some* $k \in \mathbb{Z}$. *Then* $5n + 3 = 10k + 3 = 10k + 2 + 1 = 2(5k + 1) + 1$. *Since* $5k + 1 \in \mathbb{Z}$, $5n + 3$ *is odd.*

7. (a) Give an example of a family of sets A_n (where $n \in \mathbb{N}$) such that $\cup_{n\in\mathbb{N}} A_n = \mathbb{R}$ and $\cap_{n\in\mathbb{N}} A_n = \mathbb{Z}$.

   Let $A_n = \cup_{i\in\mathbb{Z}} \left(i - \frac{1}{n}, i + \frac{1}{n}\right)$.
   Then $A_1 = \cup_{i\in\mathbb{Z}} (i - 1, i + 1) = \mathbb{R}$,
   $A_2 = \cup_{i\in\mathbb{Z}} \left(i - \frac{1}{2}, i + \frac{1}{2}\right)$,
   $A_3 = \cup_{i\in\mathbb{Z}} \left(i - \frac{1}{3}, i + \frac{1}{3}\right)$,
   $A_4 = \cup_{i\in\mathbb{Z}} \left(i - \frac{1}{4}, i + \frac{1}{4}\right)$,
   $A_5 = \cup_{i\in\mathbb{Z}} \left(i - \frac{1}{5}, i + \frac{1}{5}\right)$,
   and so on.

   All these sets are subsets of $\mathbb{R}$, *so their union is a subset of* $\mathbb{R}$. *Since* $A_1 = \mathbb{R}$, *the union is* $\mathbb{R}$.

   Each set is the union of open intervals containing integers, and these intervals become shorter and shorter. Therefore their intersection contains all integers, but no other numbers.

   (Note: show each of the above five sets on the real number line to help you see what they are.)

   (b) What are $\cup_{n=3}^{5} A_n$ and $\cap_{n=3}^{5} A_n$ for your sets?

   Notice that $A_5 \subset A_4 \subset A_3$. *Therefore*

$$\cup_{n=3}^{5} A_n = A_3 \cup A_4 \cup A_5 = A_3 = \cup_{i\in\mathbb{Z}} \left(i - \frac{1}{3}, i + \frac{1}{3}\right)$$

   and

$$\cap_{n=3}^{5} A_n = A_3 \cap A_4 \cap A_5 = A_5 = \cup_{i\in\mathbb{Z}} \left(i - \frac{1}{5}, i + \frac{1}{5}\right).$$