Test 2, extra credit problem.

Prove that for any positive integer n,

$$1 < \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n+1} < 2.$$

Proof.

First we will try to estimate the sum by estimating each term. We see that each term is between $\frac{1}{3n+1}$ and $\frac{1}{n+1}$, and there are $2n+1$ terms, therefore the sum is between $\frac{2n+1}{3n+1}$ and $\frac{2n+1}{n+1}$. The lower bound doesn't help us, but from the upper bound we see that the sum is less than $\frac{2n+2}{n+1} = 2$.

To show that the sum is bigger than 1, we will use Mathematical induction.

Basis step. For $n = 1$ we have to check that $1 < \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$. We calculate $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$, and we see that this is indeed bigger than 1.

Inductive step. Assume the inequality holds for $n = k$, i.e.

$$1 < \frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{3k+1}. \qquad (1)$$

We want to prove that it holds for $n = k+1$:

$$1 < \frac{1}{(k+1)+1} + \frac{1}{(k+1)+2} + \ldots + \frac{1}{3(k+1)+1},$$

or

$$1 < \frac{1}{k+2} + \frac{1}{k+3} + \ldots + \frac{1}{3k+1} + \frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4}. \qquad (2)$$

Compare (1) and (2), and notice that we "lost" the term $\frac{1}{k+1}$ but "gained" 3 terms: $\frac{1}{3k+2}$, $\frac{1}{3k+3}$, and $\frac{1}{3k+4}$. If we can show that we gained more than we lost, then the new sum (for $k+1$) is bigger than 1. Thus we want to show that

$$\frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} > \frac{1}{k+1}.$$

The following inequalities are equivalent:

$$\frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} > \frac{3}{3k+3}$$

$$\frac{1}{3k+2} + \frac{1}{3k+4} > \frac{2}{3k+3}$$

$$\frac{6k+6}{(3k+2)(3k+4)} > \frac{2}{3k+3}$$

$$\frac{3k+3}{(3k+2)(3k+4)} > \frac{1}{3k+3}$$

$$(3k+3)^2 > (3k+2)(3k+4)$$

$$9k^2 + 18k + 9 > 9k^2 + 18k + 8,$$

and the last one is obviously true.